Discrete Mathematics in Computer Science
E5. Syntax and Semantics of Predicate Logic

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E5.1 Syntax of Predicate Logic

E5.2 Semantics of Predicate Logic

E5.1 Syntax of Predicate Logic

Limits of Propositional Logic

Cannot well be expressed in propositional logic:

► "Everyone who does the exercises passes the exam."

► "If someone with administrator privileges presses 'delete',
   all data is gone."

► "Everyone has a mother."

► "If someone is the father of some person,
   the person is his child."

▷ need more expressive logic
   ⤳ predicate logic (a.k.a. first-order logic)

German: Prädikatenlogik (erster Stufe)

Syntax: Building Blocks

▶ **Signatures** define allowed symbols.
  analogy: atom set A in propositional logic

▶ **Terms** are associated with objects by the semantics.
  no analogy in propositional logic

▶ **Formulas** are associated with truth values (true or false)
  by the semantics.
  analogy: formulas in propositional logic

German: Signatur, Term, Formel

Signatures: Definition

Definition (Signature)

A signature (of predicate logic) is a 4-tuple $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$
consisting of the following four disjoint sets:

▶ a finite or countable set $\mathcal{V}$ of variable symbols

▶ a finite or countable set $\mathcal{C}$ of constant symbols

▶ a finite or countable set $\mathcal{F}$ of function symbols

▶ a finite or countable set $\mathcal{P}$ of predicate symbols
   (or relation symbols)

Every function symbol $f \in \mathcal{F}$ and predicate symbol $P \in \mathcal{P}$
has an associated arity $ar(f), ar(P) \in \mathbb{N}_1$ (number of arguments).

German: Variablen-, Konstanten-, Funktions-, Prädikat- und
Relationssymbole; Stelligkeit

Signatures: Terminology and Conventions

terminology:

▶ k-ary (function or predicate) symbol:
   symbol s with arity $ar(s) = k$.

▶ also: unary, binary, ternary

German: k-stellig, unär, binär, ternär

conventions (in this course):

▶ variable symbols written in *italics*,
   other symbols upright.

▶ predicate symbols begin with capital letter,
   other symbols with lower-case letters

Signatures: Examples

> ### Example: Arithmetic
> - $\mathcal{V} = \{x, y, z, x_1, x_2, x_3, \dots\}$
> - $\mathcal{C} = \{\text{zero}, \text{one}\}$
> - $\mathcal{F} = \{\text{sum}, \text{product}\}$
> - $\mathcal{P} = \{\text{Positive}, \text{SquareNumber}\}$
>
> $ar(\text{sum}) = ar(\text{product}) = 2$, $ar(\text{Positive}) = ar(\text{SquareNumber}) = 1$

Signatures: Examples

> ### Example: Genealogy
>
> ▶ $\mathcal{V} = \{x, y, z, x_1, x_2, x_3, \dots\}$
>
> ▶ $\mathcal{C} = \{\text{roger-federer}, \text{lisa-simpson}\}$
>
> ▶ $\mathcal{F} = \emptyset$
>
> ▶ $\mathcal{P} = \{\text{Female}, \text{Male}, \text{Parent}\}$
>
> $ar(\text{Female}) = ar(\text{Male}) = 1$, $ar(\text{Parent}) = 2$

Terms: Definition

Definition (Term)

Let $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ be a signature.
A term (over $\mathcal{S}$) is inductively constructed
according to the following rules:

- ▶ Every variable symbol $v \in \mathcal{V}$ is a term.
- ▶ Every constant symbol $c \in \mathcal{C}$ is a term.
- ▶ If $t_1, \ldots, t_k$ are terms and $f \in \mathcal{F}$ is a function symbol
  with arity k, then $f(t_1, \ldots, t_k)$ is a term.

German: Term

examples:

- ▶ x_4
- ▶ lisa-simpson
- ▶ $\text{sum}(x_3, \text{product}(\text{one}, x_5))$

Formulas: Definition

> ### Definition (Formula)
> For a signature $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ the set of predicate logic formulas
> (over $\mathcal{S}$) is inductively defined as follows:
>
> ▶ If $t_1, \ldots, t_k$ are terms (over $\mathcal{S}$) and $P \in \mathcal{P}$ is a k-ary predicate
>   symbol, then the atomic formula (or the atom) $P(t_1, \ldots, t_k)$
>   is a formula over $\mathcal{S}$.
>
> ▶ If t_1 and t_2 are terms (over $\mathcal{S}$), then the identity $(t_1 = t_2)$
>   is a formula over $\mathcal{S}$.
>
> ▶ If $x \in \mathcal{V}$ is a variable symbol and φ a formula over $\mathcal{S}$,
>   then the universal quantification $\forall x\, \varphi$
>   and the existential quantification $\exists x\, \varphi$ are formulas over $\mathcal{S}$.
>
> . . .

German: atomare Formel, Atom, Identität,
                Allquantifizierung, Existenzquantifizierung

Formulas: Definition

> ### Definition (Formula)
> For a signature $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ the set of predicate logic formulas
> (over $\mathcal{S}$) is inductively defined as follows:
> ...
>
> ▶ If φ is a formula over $\mathcal{S}$, then so is its negation $\neg\varphi$.
>
> ▶ If φ and ψ are formulas over $\mathcal{S}$, then so are
> the conjunction $(\varphi \wedge \psi)$ and the disjunction $(\varphi \vee \psi)$.

German: Negation, Konjunktion, Disjunktion

Formulas: Examples

Examples: Arithmetic and Genealogy

▶ $\text{Positive}(x_2)$

▶ $\forall x \, (\neg \text{SquareNumber}(x) \vee \text{Positive}(x))$

▶ $\exists x_3 \, (\text{SquareNumber}(x_3) \wedge \neg \text{Positive}(x_3))$

▶ $\forall x \, (x = y)$

▶ $\forall x \, (\text{sum}(x, x) = \text{product}(x, \text{one}))$

▶ $\forall x \exists y \, (\text{sum}(x, y) = \text{zero})$

▶ $\forall x \exists y \, (\text{Parent}(y, x) \wedge \text{Female}(y))$

Terminology: The symbols $\forall$ and $\exists$ are called quantifiers.

German: Quantoren

Abbreviations and Placement of Parentheses by Convention

abbreviations:

- ▶ $(\varphi \to \psi)$ is an abbreviation for $(\neg\varphi \lor \psi)$.
- ▶ $(\varphi \leftrightarrow \psi)$ is an abbreviation for $((\varphi \to \psi) \land (\psi \to \varphi))$.
- ▶ Sequences of the same quantifier can be abbreviated.
  For example:
  - ▶ $\forall x \forall y \forall z\, \varphi \rightsquigarrow \forall xyz\, \varphi$
  - ▶ $\exists x \exists y \exists z\, \varphi \rightsquigarrow \exists xyz\, \varphi$
  - ▶ $\forall w \exists x \exists y \forall z\, \varphi \rightsquigarrow \forall w \exists xy \forall z\, \varphi$

placement of parentheses by convention:

- ▶ analogous to propositional logic
- ▶ quantifiers $\forall$ and $\exists$ bind more strongly than anything else.
- ▶ example: $\forall x\, P(x) \to Q(x)$ corresponds to $(\forall x\, P(x) \to Q(x))$,
           not $\forall x\, (P(x) \to Q(x))$.

Exercise

$\mathcal{S} = \langle \{x, y, z\}, \{c\}, \{f, g, h\}, \{Q, R, S\} \rangle$ with
$ar(f) = 3, ar(g) = ar(h) = 1, ar(Q) = 2, ar(R) = ar(S) = 1$

▶ $f(x, y)$

▶ $(g(x) = R(y))$

▶ $(g(x) = f(y, c, h(x)))$

▶ $(R(x) \land \forall x\, S(x))$

▶ $\forall c\, Q(c, x)$

▶ $(\forall x \exists y\, (g(x) = y) \lor (h(x) = c))$

Which expressions are syntactically correct formulas or terms for $\mathcal{S}$?
What kind of term/formula?

E5.2 Semantics of Predicate Logic

Semantics: Motivation

▶ interpretations in propositional logic:
  truth assignments for the propositional variables

▶ There are no propositional variables in predicate logic.

▶ instead: interpretation determines meaning
  of the constant, function and predicate symbols.

▶ meaning of variable symbols not determined by interpretation
  but by separate variable assignment

Interpretations and Variable Assignments

Let $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ be a signature.

Definition (Interpretation, Variable Assignment)

An **interpretation** (for $\mathcal{S}$) is a pair $\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ of:

- ▶ a non-empty set U called the **universe** and
- ▶ a function $\cdot^{\mathcal{I}}$ that assigns a meaning to the constant, function, and predicate symbols:
    - ▶ $c^{\mathcal{I}} \in U$ for constant symbols $c \in \mathcal{C}$
    - ▶ $f^{\mathcal{I}} : U^k \to U$ for k-ary function symbols $f \in \mathcal{F}$
    - ▶ $P^{\mathcal{I}} \subseteq U^k$ for k-ary predicate symbols $P \in \mathcal{P}$

A **variable assignment** (for $\mathcal{S}$ and universe U)
is a function $\alpha : \mathcal{V} \to U$.

German: Interpretation, Universum (or Grundmenge),
          Variablenzuweisung

Interpretations and Variable Assignments: Example

Example

signature: $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ with $\mathcal{V} = \{x, y, z\}$,
$\mathcal{C} = \{\text{zero}, \text{one}\}$, $\mathcal{F} = \{\text{sum}, \text{product}\}$, $\mathcal{P} = \{\text{SquareNumber}\}$
$ar(\text{sum}) = ar(\text{product}) = 2$, $ar(\text{SquareNumber}) = 1$

$\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ with

- $U = \{u_0, u_1, u_2, u_3, u_4, u_5, u_6\}$
- $\text{zero}^{\mathcal{I}} = u_0$
- $\text{one}^{\mathcal{I}} = u_1$
- $\text{sum}^{\mathcal{I}}(u_i, u_j) = u_{(i+j) \bmod 7}$ for all $i, j \in \{0, \ldots, 6\}$
- $\text{product}^{\mathcal{I}}(u_i, u_j) = u_{(i \cdot j) \bmod 7}$ for all $i, j \in \{0, \ldots, 6\}$
- $\text{SquareNumber}^{\mathcal{I}} = \{u_0, u_1, u_2, u_4\}$

$\alpha = \{x \mapsto u_5, y \mapsto u_5, z \mapsto u_0\}$

Semantics: Informally

Example: $(\forall x(\text{Block}(x) \rightarrow \text{Red}(x)) \wedge \text{Block}(a))$
"For all objects x: if x is a block, then x is red.
Also, the object called a is a block."

▶ Terms are interpreted as objects.

▶ Unary predicates denote properties of objects
   (to be a block, to be red, to be a square number, . . .).

▶ General predicates denote relations between objects
   (to be someone's child, to have a common divisor, . . .).

▶ Universally quantified formulas ("$\forall$") are true
   if they hold for every object in the universe.

▶ Existentially quantified formulas ("$\exists$") are true
   if they hold for at least one object in the universe.

Interpretations of Terms

Let $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ be a signature.

Definition (Interpretation of a Term)

Let $\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ be an interpretation for $\mathcal{S}$,
and let α be a variable assignment for $\mathcal{S}$ and universe U.

Let t be a term over $\mathcal{S}$.
The interpretation of t under $\mathcal{I}$ and α, written as $t^{\mathcal{I},\alpha}$,
is the element of the universe U defined as follows:

- If $t = x$ with $x \in \mathcal{V}$ (t is a variable term):
  $x^{\mathcal{I},\alpha} = \alpha(x)$

- If $t = c$ with $c \in \mathcal{C}$ (t is a constant term):
  $c^{\mathcal{I},\alpha} = c^{\mathcal{I}}$

- If $t = f(t_1, \ldots, t_k)$ (t is a function term):
  $f(t_1, \ldots, t_k)^{\mathcal{I},\alpha} = f^{\mathcal{I}}(t_1^{\mathcal{I},\alpha}, \ldots, t_k^{\mathcal{I},\alpha})$

Interpretations of Terms: Example

Example

signature: $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$

with $\mathcal{V} = \{x, y, z\}$, $\mathcal{C} = \{\text{zero}, \text{one}\}$, $\mathcal{F} = \{\text{sum}, \text{product}\}$,

$ar(\text{sum}) = ar(\text{product}) = 2$

$\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ with

▶ $U = \{u_0, u_1, u_2, u_3, u_4, u_5, u_6\}$

▶ $\text{zero}^{\mathcal{I}} = u_0$

▶ $\text{one}^{\mathcal{I}} = u_1$

▶ $\text{sum}^{\mathcal{I}}(u_i, u_j) = u_{(i+j) \bmod 7}$ for all $i, j \in \{0, \ldots, 6\}$

▶ $\text{product}^{\mathcal{I}}(u_i, u_j) = u_{(i \cdot j) \bmod 7}$ for all $i, j \in \{0, \ldots, 6\}$

$\alpha = \{x \mapsto u_5, y \mapsto u_5, z \mapsto u_0\}$

Interpretations of Terms: Example (ctd.)

> ### Example (ctd.)
> ▶ $\text{zero}^{\mathcal{I},\alpha} =$
>
>
> ▶ $y^{\mathcal{I},\alpha} =$
>
>
> ▶ $\text{sum}(x, y)^{\mathcal{I},\alpha} =$
>
>
> ▶ $\text{product}(\text{one}, \text{sum}(x, \text{zero}))^{\mathcal{I},\alpha} =$

Semantics of Predicate Logic Formulas

Let $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ be a signature.

Definition (Formula is Satisfied or True)

Let $\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ be an interpretation for $\mathcal{S}$,
and let α be a variable assignment for $\mathcal{S}$ and universe U.
We say that $\mathcal{I}$ and α satisfy a predicate logic formula φ
(also: φ is true under $\mathcal{I}$ and α), written: $\mathcal{I}, \alpha \models \varphi$,
according to the following inductive rules:

$$\mathcal{I}, \alpha \models \mathsf{P}(t_1, \ldots, t_k) \quad \text{iff } \langle t_1^{\mathcal{I},\alpha}, \ldots, t_k^{\mathcal{I},\alpha} \rangle \in \mathsf{P}^{\mathcal{I}}$$

$$\mathcal{I}, \alpha \models (t_1 = t_2) \quad \text{iff } t_1^{\mathcal{I},\alpha} = t_2^{\mathcal{I},\alpha}$$

$$\mathcal{I}, \alpha \models \neg\varphi \quad \text{iff } \mathcal{I}, \alpha \not\models \varphi$$

$$\mathcal{I}, \alpha \models (\varphi \wedge \psi) \quad \text{iff } \mathcal{I}, \alpha \models \varphi \text{ and } \mathcal{I}, \alpha \models \psi$$

$$\mathcal{I}, \alpha \models (\varphi \vee \psi) \quad \text{iff } \mathcal{I}, \alpha \models \varphi \text{ or } \mathcal{I}, \alpha \models \psi \qquad \ldots$$

German: $\mathcal{I}$ und α erfüllen φ (also: φ ist wahr unter $\mathcal{I}$ und α)

Semantics of Predicate Logic Formulas

Let $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ be a signature.

Definition (Formula is Satisfied or True)

$\cdots$

$$\mathcal{I}, \alpha \models \forall x \varphi \quad \text{iff } \mathcal{I}, \alpha[x := u] \models \varphi \text{ for all } u \in U$$

$$\mathcal{I}, \alpha \models \exists x \varphi \quad \text{iff } \mathcal{I}, \alpha[x := u] \models \varphi \text{ for at least one } u \in U$$

where $\alpha[x := u]$ is the same variable assignment as α,
except that it maps variable x to the value u.
Formally:

$$(\alpha[x := u])(z) = \begin{cases} u & \text{if } z = x \\ \alpha(z) & \text{if } z \neq x \end{cases}$$

Semantics: Example

Example

signature: $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$
with $\mathcal{V} = \{x, y, z\}$, $\mathcal{C} = \{a, b\}$, $\mathcal{F} = \emptyset$, $\mathcal{P} = \{\text{Block}, \text{Red}\}$,
$ar(\text{Block}) = ar(\text{Red}) = 1$.

$\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ with

▶ $U = \{u_1, u_2, u_3, u_4, u_5\}$

▶ $a^{\mathcal{I}} = u_1$

▶ $b^{\mathcal{I}} = u_3$

▶ $\text{Block}^{\mathcal{I}} = \{u_1, u_2\}$

▶ $\text{Red}^{\mathcal{I}} = \{u_1, u_2, u_3, u_5\}$

$\alpha = \{x \mapsto u_1, y \mapsto u_2, z \mapsto u_1\}$

Semantics: Example (ctd.)

Example (ctd.)

Questions:

- $\mathcal{I}, \alpha \models (\text{Block}(b) \lor \neg\text{Block}(b))$?
- $\mathcal{I}, \alpha \models (\text{Block}(x) \rightarrow (\text{Block}(x) \lor \neg\text{Block}(y)))$?
- $\mathcal{I}, \alpha \models (\text{Block}(a) \land \text{Block}(b))$?
- $\mathcal{I}, \alpha \models \forall x(\text{Block}(x) \rightarrow \text{Red}(x))$?