# Discrete Mathematics in Computer Science

## E2. Properties of Formulas and Equivalences

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E2.1 Properties of Propositional Formulas

E2.2 Equivalences

# E2.1 Properties of Propositional Formulas

## The Story So Far

▶ propositional logic based on atomic propositions

▶ syntax: which formulas are well-formed?

▶ semantics: when is a formula true?

▶ interpretations: important basis of semantics

# Reminder: Syntax of Propositional Logic

**Definition (Syntax of Propositional Logic)**

Let $A$ be a set of atomic propositions. The set of propositional formulas (over $A$) is inductively defined as follows:

- ▶ Every atom $a \in A$ is a propositional formula over $A$.

- ▶ If $\varphi$ is a propositional formula over $A$,
  then so is its negation $\neg\varphi$.

- ▶ If $\varphi$ and $\psi$ are propositional formulas over $A$,
  then so is the conjunction $(\varphi \wedge \psi)$.

- ▶ If $\varphi$ and $\psi$ are propositional formulas over $A$,
  then so is the disjunction $(\varphi \vee \psi)$.

The implication $(\varphi \rightarrow \psi)$ is an abbreviation for $(\neg\varphi \vee \psi)$.
The biconditional $(\varphi \leftrightarrow \psi)$ is an abbrev. for $((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi))$.

# Reminder: Semantics of Propositional Logic

Definition (Semantics of Propositional Logic)

A truth assignment (or interpretation) for a set of atomic propositions $A$ is a function $\mathcal{I} : A \to \{0, 1\}$.

A propositional formula $\varphi$ (over $A$) holds under $\mathcal{I}$ (written as $\mathcal{I} \models \varphi$) according to the following definition:

$$\mathcal{I} \models a \qquad \text{iff} \quad \mathcal{I}(a) = 1 \qquad\qquad (\text{for } a \in A)$$
$$\mathcal{I} \models \neg\varphi \qquad \text{iff} \quad \text{not } \mathcal{I} \models \varphi$$
$$\mathcal{I} \models (\varphi \wedge \psi) \quad \text{iff} \quad \mathcal{I} \models \varphi \text{ and } \mathcal{I} \models \psi$$
$$\mathcal{I} \models (\varphi \vee \psi) \quad \text{iff} \quad \mathcal{I} \models \varphi \text{ or } \mathcal{I} \models \psi$$

## Properties of Propositional Formulas

A propositional formula $\varphi$ is

- ▶ satisfiable if $\varphi$ has at least one model
- ▶ unsatisfiable if $\varphi$ is not satisfiable
- ▶ valid (or a tautology) if $\varphi$ is true under every interpretation
- ▶ falsifiable if $\varphi$ is no tautology

German: erfüllbar, unerfüllbar, allgemeingültig/eine Tautologie, falsifizierbar

## Examples

How can we show that a formula has one of these properties?

▶ Show that $(A \wedge B)$ is satisfiable.
  $\mathcal{I} = \{A \mapsto 1, B \mapsto 1\}$  (+ simple proof that $\mathcal{I} \models (A \wedge B)$)

▶ Show that $(A \wedge B)$ is falsifiable.
  $\mathcal{I} = \{A \mapsto 0, B \mapsto 1\}$  (+ simple proof that $\mathcal{I} \not\models (A \wedge B)$)

▶ Show that $(A \wedge B)$ is not valid.
  Follows directly from falsifiability.

▶ Show that $(A \wedge B)$ is not unsatisfiable.
  Follows directly from satisfiability.

So far all proofs by specifying one interpretation.

How to prove that a given formula is valid/unsatisfiable/
not satisfiable/not falsifiable?

⤳ must consider all possible interpretations

## Truth Tables

Evaluate for all possible interpretations
if they are models of the considered formula.

| $\mathcal{I}(A)$ | $\mathcal{I} \models \neg A$ |
| --- | --- |
| 0 | Yes |
| 1 | No |

| $\mathcal{I}(A)$ | $\mathcal{I}(B)$ | $\mathcal{I} \models (A \wedge B)$ |
| --- | --- | --- |
| 0 | 0 | No |
| 0 | 1 | No |
| 1 | 0 | No |
| 1 | 1 | Yes |

| $\mathcal{I}(A)$ | $\mathcal{I}(B)$ | $\mathcal{I} \models (A \vee B)$ |
| --- | --- | --- |
| 0 | 0 | No |
| 0 | 1 | Yes |
| 1 | 0 | Yes |
| 1 | 1 | Yes |

## Truth Tables in General

Similarly in the case where we consider a formula whose building
blocks are themselves arbitrary unspecified formulas:

| $\mathcal{I} \models \varphi$ | $\mathcal{I} \models \psi$ | $\mathcal{I} \models (\varphi \wedge \psi)$ |
| :---: | :---: | :---: |
| No | No | No |
| No | Yes | No |
| Yes | No | No |
| Yes | Yes | Yes |

## Truth Tables for Properties of Formulas

Is $\varphi = ((A \to B) \vee (\neg B \to A))$ valid, unsatisfiable, ...?

| $\mathcal{I}(A)$ | $\mathcal{I}(B)$ | $\mathcal{I} \models \neg B$ | $\mathcal{I} \models (A \to B)$ | $\mathcal{I} \models (\neg B \to A)$ | $\mathcal{I} \models \varphi$ |
|---|---|---|---|---|---|
| 0 | 0 | Yes | Yes | No | Yes |
| 0 | 1 | No | Yes | Yes | Yes |
| 1 | 0 | Yes | No | Yes | Yes |
| 1 | 1 | No | Yes | Yes | Yes |

# Connection Between Formula Properties and Truth Tables

A propositional formula $\varphi$ is

▶ satisfiable if $\varphi$ has at least one model
   ⤳ result in at least one row is "Yes"

▶ unsatisfiable if $\varphi$ is not satisfiable
   ⤳ result in all rows is "No"

▶ valid (or a tautology) if $\varphi$ is true under every interpretation
   ⤳ result in all rows is "Yes"

▶ falsifiable if $\varphi$ is no tautology
   ⤳ result in at least one row is "No"

## Main Disadvantage of Truth Tables

How big is a truth table with $n$ atomic propositions?

| 1 | 2 interpretations (rows) |
|---|---|
| 2 | 4 interpretations (rows) |
| 3 | 8 interpretations (rows) |
| $n$ | ??? interpretations |

Some examples: $2^{10} = 1024$, $2^{20} = 1048576$, $2^{30} = 1073741824$

⤳ not viable for larger formulas; we need a different solution

▶ more on difficulty of satisfiability etc.:
  Theory of Computer Science course

▶ practical algorithms: Foundations of AI course

# E2.2 Equivalences

# Equivalent Formulas

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**Definition (Equivalence of Propositional Formulas)**

Two propositional formulas $\varphi$ and $\psi$ over $A$ are (logically) equivalent ($\varphi \equiv \psi$) if for all interpretations $\mathcal{I}$ for $A$ it is true that $\mathcal{I} \models \varphi$ if and only if $\mathcal{I} \models \psi$.

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German: logisch äquivalent

## Equivalent Formulas: Example

$$((\varphi \vee \psi) \vee \chi) \equiv (\varphi \vee (\psi \vee \chi))$$

| $\mathcal{I} \models$ $\varphi$ | $\mathcal{I} \models$ $\psi$ | $\mathcal{I} \models$ $\chi$ | $\mathcal{I} \models$ $(\varphi \vee \psi)$ | $\mathcal{I} \models$ $(\psi \vee \chi)$ | $\mathcal{I} \models$ $((\varphi \vee \psi) \vee \chi)$ | $\mathcal{I} \models$ $(\varphi \vee (\psi \vee \chi))$ |
|---|---|---|---|---|---|---|
| No | No | No | No | No | No | No |
| No | No | Yes | No | Yes | Yes | Yes |
| No | Yes | No | Yes | Yes | Yes | Yes |
| No | Yes | Yes | Yes | Yes | Yes | Yes |
| Yes | No | No | Yes | No | Yes | Yes |
| Yes | No | Yes | Yes | Yes | Yes | Yes |
| Yes | Yes | No | Yes | Yes | Yes | Yes |
| Yes | Yes | Yes | Yes | Yes | Yes | Yes |

## Some Equivalences (1)

$$(\varphi \wedge \varphi) \equiv \varphi$$
$$(\varphi \vee \varphi) \equiv \varphi \qquad \text{(idempotence)}$$

$$(\varphi \wedge \psi) \equiv (\psi \wedge \varphi)$$
$$(\varphi \vee \psi) \equiv (\psi \vee \varphi) \qquad \text{(commutativity)}$$

$$((\varphi \wedge \psi) \wedge \chi) \equiv (\varphi \wedge (\psi \wedge \chi))$$
$$((\varphi \vee \psi) \vee \chi) \equiv (\varphi \vee (\psi \vee \chi)) \quad \text{(associativity)}$$

German: Idempotenz, Kommutativität, Assoziativität

## Some Equivalences (2)

$$(\varphi \wedge (\varphi \vee \psi)) \equiv \varphi$$
$$(\varphi \vee (\varphi \wedge \psi)) \equiv \varphi \qquad \text{(absorption)}$$
$$(\varphi \wedge (\psi \vee \chi)) \equiv ((\varphi \wedge \psi) \vee (\varphi \wedge \chi))$$
$$(\varphi \vee (\psi \wedge \chi)) \equiv ((\varphi \vee \psi) \wedge (\varphi \vee \chi)) \quad \text{(distributivity)}$$

German: Absorption, Distributivität

## Some Equivalences (3)

$$\neg\neg\varphi \equiv \varphi \qquad \text{(Double negation)}$$

$$\neg(\varphi \wedge \psi) \equiv (\neg\varphi \vee \neg\psi)$$
$$\neg(\varphi \vee \psi) \equiv (\neg\varphi \wedge \neg\psi) \qquad \text{(De Morgan's rules)}$$

$$(\varphi \vee \psi) \equiv \varphi \text{ if } \varphi \text{ tautology}$$
$$(\varphi \wedge \psi) \equiv \psi \text{ if } \varphi \text{ tautology} \qquad \text{(tautology rules)}$$

$$(\varphi \vee \psi) \equiv \psi \text{ if } \varphi \text{ unsatisfiable}$$
$$(\varphi \wedge \psi) \equiv \varphi \text{ if } \varphi \text{ unsatisfiable} \qquad \text{(unsatisfiability rules)}$$

German: Doppelnegation, De Morgansche Regeln,
Tautologieregeln, Unerfüllbarkeitsregeln

## Substitution Theorem

> **Theorem (Substitution Theorem)**
>
> *Let $\varphi$ and $\varphi'$ be equivalent propositional formulas over A.*
> *Let $\psi$ be a propositional formula with (at least)*
> *one occurrence of the subformula $\varphi$.*
>
> *Then $\psi$ is equivalent to $\psi'$, where $\psi'$ is constructed from $\psi$*
> *by replacing an occurrence of $\varphi$ in $\psi$ with $\varphi'$.*

German: Ersetzbarkeitstheorem

(without proof)

## Application of Equivalences: Example

$$
\begin{aligned}
(P \wedge (Q \vee \neg P)) &\equiv ((P \wedge Q) \vee (P \wedge \neg P)) && \text{(distributivity)} \\
&\equiv ((P \wedge \neg P) \vee (P \wedge Q)) && \text{(commutativity)} \\
&\equiv (P \wedge Q) && \text{(unsatisfiability rule)}
\end{aligned}
$$