Discrete Mathematics in Computer Science
E1. Syntax and Semantics of Propositional Logic

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E1.1 Introduction to Formal Logic

E1.2 Syntax of Propositional Logic

E1.3 Semantics of Propositional Logic

E1.1 Introduction to Formal Logic

Why Logic?

- formalizing mathematics
  - What is a true statement?
  - What is a valid proof?
- basis of many tools in computer science
  - design of digital circuits
  - semantics of databases; query optimization
  - meaning of programming languages
  - verification of safety-critical hardware/software
  - knowledge representation in artificial intelligence
  - logic-based programming languages (e.g. Prolog)
  - . . .

Application: Logic Programming I

Declarative approach: Describe what to accomplish,
not how to accomplish it.

Example (Map Coloring)
Color each region in a map with a limited number of colors
so that no two adjacent regions have the same color.

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This is a hard problem!

Application: Logic Programming II

Prolog program

```
color(red). color(blue). color(green). color(yellow).

neighbor(StateAColor, StateBColor) :-
    color(StateAColor), color(StateBColor),
    StateAColor \= StateBColor.

switzerland(AG, AI, AR, BE, BL, BS, FR, GE, GL, GR,
            JU, LU, NE, NW, OW, SG, SH, SO, SZ, TG,
            TI, UR, VD, VS, ZG, ZH) :-
    neighbor(AG, BE), neighbor(AG, BL), neighbor(AG, LU),
    ...
    neighbor(UR, VS), neighbor(VD, VS), neighbor(ZH, ZG).
```

What Logic is About

General Question:
- ▶ Given some knowledge about the world (a knowledge base)
- ▶ what can we derive from it?
- ▶ And on what basis may we argue?

⤳ logic

Goal: "mechanical" proofs
- ▶ formal "game with letters"
- ▶ detached from a concrete meaning

Task

What's the secret of your long life?

I am on a strict diet: If I don't drink beer to a meal, then I always eat fish. Whenever I have fish and beer with the same meal, I abstain from ice cream. When I eat ice cream or don't drink beer, then I never touch fish.

Simplify this advice!

Exercise from U. Schöning: Logik für Informatiker
Picture courtesy of graur razvan ionut / FreeDigitalPhotos.net

Propositional Logic

Propositional logic is a simple logic without numbers or objects.

Building blocks of propositional logic:

- ▶ propositions are statements that can be either true or false
- ▶ atomic propositions cannot be split into sub-propositions
- ▶ logical connectives connect propositions to form new ones

German: Aussagenlogik, Aussage, atomare Aussage, Junktoren

Examples for Building Blocks

If I don't drink beer to a meal, then I always eat fish. Whenever I have fish and beer with the same meal, I abstain from ice cream. When I eat ice cream or don't drink beer, then I never touch fish.

- ▶ Every sentence is a proposition that consists of sub-propositions (e. g., "eat ice cream or don't drink beer").
- ▶ atomic propositions "drink beer", "eat fish", "eat ice cream"
- ▶ logical connectives "and", "or", negation, "if, then"

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Problems with Natural Language

If I don't drink beer to a meal, then I always eat fish.
Whenever I have fish and beer with the same meal, I abstain from ice cream. When I eat ice cream or don't drink beer, then I never touch fish.

- ▶ "irrelevant" information

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Problems with Natural Language

If I don't drink beer, then I eat fish.
Whenever I have fish and beer, I abstain from ice cream.
When I eat ice cream or don't drink beer, then I never touch fish.

► "irrelevant" information
► different formulations for the same connective/proposition

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Picture courtesy of graur razvan ionut / FreeDigitalPhotos.net

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Problems with Natural Language

If not DrinkBeer, then EatFish.
If EatFish and DrinkBeer,
then not EatIceCream.
If EatIceCream or not DrinkBeer,
then not EatFish.

► "irrelevant" information
► different formulations for the same connective/proposition

Exercise from U. Schöning: Logik für Informatiker
Picture courtesy of graur razvan ionut / FreeDigitalPhotos.net

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What is Next?

► What are meaningful (well-defined) sequences of
  atomic propositions and connectives?
  "if then EatIceCream not or DrinkBeer and" not meaningful
  → syntax
► What does it mean if we say that a statement is true?
  Is "DrinkBeer and EatFish" true?
  → semantics
► When does a statement logically follow from another?
  Does "EatFish" follow from "if DrinkBeer, then EatFish"?
  → logical entailment

German: Syntax, Semantik, logische Folgerung

E1.2 Syntax of Propositional Logic

Syntax of Propositional Logic

> **Definition (Syntax of Propositional Logic)**
>
> Let A be a set of atomic propositions. The set of propositional formulas (over A) is inductively defined as follows:
>
> - Every atom $a \in A$ is a propositional formula over A.
> - If φ is a propositional formula over A, then so is its negation $\neg\varphi$.
> - If φ and ψ are propositional formulas over A, then so is the conjunction $(\varphi \wedge \psi)$.
> - If φ and ψ are propositional formulas over A, then so is the disjunction $(\varphi \vee \psi)$.

The implication $(\varphi \to \psi)$ is an abbreviation for $(\neg\varphi \vee \psi)$.
The biconditional $(\varphi \leftrightarrow \psi)$ is an abbrev. for $((\varphi \to \psi) \wedge (\psi \to \varphi))$.
German: atomare Aussage, aussagenlogische Formel, Atom, Negation, Konjunktion, Disjunktion, Implikation, Bikonditional

Syntax: Examples

Which of the following sequences of symbols are propositional formulas over the set of all possible letter sequences? Which kinds of formula are they (atom, conjunction, . . .)?

- $(A \wedge (B \vee C))$
- $((\text{EatFish} \wedge \text{DrinkBeer}) \to \neg\text{EatIceCream})$
- $\neg(\wedge \text{Rain} \vee \text{StreetWet})$
- $\neg(\text{Rain} \vee \text{StreetWet})$
- $\neg(A = B)$
- $(A \wedge \neg(B \leftrightarrow)C)$
- $(A \vee \neg(B \leftrightarrow C))$
- $((A \leq B) \wedge C)$
- $((A_1 \wedge A_2) \vee \neg(A_3 \leftrightarrow A_2))$

E1.3 Semantics of Propositional Logic

Meaning of Propositional Formulas?

So far propositional formulas are only symbol sequences without any meaning.

For example, what does this mean:
$((\text{EatFish} \wedge \text{DrinkBeer}) \to \neg\text{EatIceCream})$?

▷ We need semantics!

Semantics of Propositional Logic

> **Definition (Semantics of Propositional Logic)**
>
> A truth assignment (or interpretation) for a set of atomic propositions A is a function $\mathcal{I} : A \to \{0,1\}$.
>
> A propositional formula φ (over A) holds under $\mathcal{I}$ (written as $\mathcal{I} \models \varphi$) according to the following definition:
>
> | | | | |
> |---|---|---|---|
> | $\mathcal{I} \models a$ | iff | $\mathcal{I}(a) = 1$ | (for $a \in A$) |
> | $\mathcal{I} \models \neg\varphi$ | iff | not $\mathcal{I} \models \varphi$ | |
> | $\mathcal{I} \models (\varphi \wedge \psi)$ | iff | $\mathcal{I} \models \varphi$ and $\mathcal{I} \models \psi$ | |
> | $\mathcal{I} \models (\varphi \vee \psi)$ | iff | $\mathcal{I} \models \varphi$ or $\mathcal{I} \models \psi$ | |

Question: should we define semantics of $(\varphi \to \psi)$ and $(\varphi \leftrightarrow \psi)$?

German: Wahrheitsbelegung/Interpretation, φ gilt unter $\mathcal{I}$

Semantics of Propositional Logic: Terminology

▶ For $\mathcal{I} \models \varphi$ we also say $\mathcal{I}$ is a model of φ and that φ is true under $\mathcal{I}$.

▶ If φ does not hold under $\mathcal{I}$, we write this as $\mathcal{I} \not\models \varphi$ and say that $\mathcal{I}$ is no model of φ and that φ is false under $\mathcal{I}$.

▶ Note: $\models$ is not part of the formula but part of the meta language (speaking about a formula).

German: $\mathcal{I}$ ist ein/kein Modell von φ; φ ist wahr/falsch unter $\mathcal{I}$; Metasprache

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Exercise

Consider set $A = \{X, Y, Z\}$ of atomic propositions and formula $\varphi = (X \wedge \neg Y)$.

Specify an interpretation $\mathcal{I}$ for A with $\mathcal{I} \models \varphi$.

Semantics: Example (1)

$$A = \{\text{DrinkBeer}, \text{EatFish}, \text{EatIceCream}\}$$
$$\mathcal{I} = \{\text{DrinkBeer} \mapsto 1, \text{EatFish} \mapsto 0, \text{EatIceCream} \mapsto 1\}$$
$$\varphi = (\neg\text{DrinkBeer} \to \text{EatFish})$$

Do we have $\mathcal{I} \models \varphi$?

Semantics: Example (2)

Goal: prove $\mathcal{I} \models \varphi$.

Let us use the definitions we have seen:

$$\mathcal{I} \models \varphi \text{ iff } \mathcal{I} \models (\neg\text{DrinkBeer} \rightarrow \text{EatFish})$$
$$\text{iff } \mathcal{I} \models (\neg\neg\text{DrinkBeer} \vee \text{EatFish})$$
$$\text{iff } \mathcal{I} \models \neg\neg\text{DrinkBeer} \text{ or } \mathcal{I} \models \text{EatFish}$$

This means that if we want to prove $\mathcal{I} \models \varphi$, it is sufficient to prove

$$\mathcal{I} \models \neg\neg\text{DrinkBeer}$$

or to prove

$$\mathcal{I} \models \text{EatFish.}$$

We attempt to prove the first of these statements.

Semantics: Example (3)

New goal: prove $\mathcal{I} \models \neg\neg\text{DrinkBeer}$.

We again use the definitions:

$$\mathcal{I} \models \neg\neg\text{DrinkBeer} \text{ iff not } \mathcal{I} \models \neg\text{DrinkBeer}$$
$$\text{iff not not } \mathcal{I} \models \text{DrinkBeer}$$
$$\text{iff } \mathcal{I} \models \text{DrinkBeer}$$
$$\text{iff } \mathcal{I}(\text{DrinkBeer}) = 1$$

The last statement is true for our interpretation $\mathcal{I}$.

To write this up as a proof of $\mathcal{I} \models \varphi$,
we can go through this line of reasoning back-to-front,
starting from our assumptions and ending with the conclusion
we want to show.

Semantics: Example (4)

Let $\mathcal{I} = \{\text{DrinkBeer} \mapsto 1, \text{EatFish} \mapsto 0, \text{EatIceCream} \mapsto 1\}$.

Proof that $\mathcal{I} \models (\neg\text{DrinkBeer} \rightarrow \text{EatFish})$:

1. We have $\mathcal{I} \models \text{DrinkBeer}$
   (uses defn. of $\models$ for atomic props. and fact
   $\mathcal{I}(\text{DrinkBeer}) = 1$).
2. From (1), we get $\mathcal{I} \not\models \neg\text{DrinkBeer}$
   (uses defn. of $\models$ for negations).
3. From (2), we get $\mathcal{I} \models \neg\neg\text{DrinkBeer}$
   (uses defn. of $\models$ for negations).
4. From (3), we get $\mathcal{I} \models (\neg\neg\text{DrinkBeer} \vee \psi)$ for all formulas ψ,
   in particular $\mathcal{I} \models (\neg\neg\text{DrinkBeer} \vee \text{EatFish})$
   (uses defn. of $\models$ for disjunctions).
5. From (4), we get $\mathcal{I} \models (\neg\text{DrinkBeer} \rightarrow \text{EatFish})$
   (uses defn. of "$\rightarrow$").   □

Summary

- propositional logic based on atomic propositions
- syntax defines what well-formed formulas are
- semantics defines when a formula is true
- interpretations are the basis of semantics