

Discrete Mathematics in Computer Science

E1. Syntax and Semantics of Propositional Logic

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— E1. Syntax and Semantics of Propositional Logic

E1.1 Introduction to Formal Logic

E1.2 Syntax of Propositional Logic

E1.3 Semantics of Propositional Logic

E1.1 Introduction to Formal Logic

Why Logic?

- ▶ formalizing mathematics
 - ▶ What is a true statement?
 - ▶ What is a valid proof?
- ▶ basis of many tools in computer science
 - ▶ design of digital circuits
 - ▶ semantics of databases; query optimization
 - ▶ meaning of programming languages
 - ▶ verification of safety-critical hardware/software
 - ▶ knowledge representation in artificial intelligence
 - ▶ logic-based programming languages (e.g. Prolog)
 - ▶ ...

Application: Logic Programming I

Declarative approach: Describe **what** to accomplish,
not how to accomplish it.

Example (Map Coloring)

Color each region in a map with a limited number of colors so that no two adjacent regions have the same color.



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This is a hard problem!

Application: Logic Programming II

Prolog program

```
color(red). color(blue). color(green). color(yellow).

neighbor(StateAColor, StateBColor) :-
    color(StateAColor), color(StateBColor),
    StateAColor \= StateBColor.

switzerland(AG, AI, AR, BE, BL, BS, FR, GE, GL, GR,
            JU, LU, NE, NW, OW, SG, SH, SO, SZ, TG,
            TI, UR, VD, VS, ZG, ZH) :-
    neighbor(AG, BE), neighbor(AG, BL), neighbor(AG, LU),
    ...
    neighbor(UR, VS), neighbor(VD, VS), neighbor(ZH, ZG).
```

What Logic is About

General Question:

- ▶ Given some knowledge about the world (a **knowledge base**)
- ▶ what can we **derive** from it?
- ▶ And on what basis may we argue?

↪ **logic**

Goal: “mechanical” proofs

- ▶ formal “game with letters”
- ▶ detached from a concrete meaning

Task

What's the secret of your long life?



I am on a strict diet: If I don't drink beer to a meal, then I always eat fish. Whenever I have fish and beer with the same meal, I abstain from ice cream. When I eat ice cream or don't drink beer, then I never touch fish.

Simplify this advice!

Exercise from U. Schöning: Logik für Informatiker
Picture courtesy of graur razvan ionut / FreeDigitalPhotos.net

Propositional Logic

Propositional logic is a simple logic without numbers or objects.

Building blocks of propositional logic:

- ▶ **propositions** are statements that can be either true or false
- ▶ **atomic propositions** cannot be split into sub-propositions
- ▶ **logical connectives** connect propositions to form new ones

German: Aussagenlogik, Aussage, atomare Aussage, Junktoren

Examples for Building Blocks



If I don't **drink beer** to a meal, then I always **eat fish**. Whenever I **have fish** and **beer** with the same meal, I abstain from **ice cream**. When I **eat ice cream** or don't **drink beer**, then I never touch **fish**.

- ▶ Every sentence is a proposition that consists of sub-propositions (e. g., “eat ice cream or don't drink beer”).
- ▶ atomic propositions “**drink beer**”, “**eat fish**”, “**eat ice cream**”
- ▶ logical connectives “and”, “or”, negation, “if, then”

Exercise from U. Schöning: Logik für Informatiker
Picture courtesy of graur razvan ionut / FreeDigitalPhotos.net

Examples for Building Blocks



If I **don't** drink beer to a meal, **then** I always eat fish. **Whenever** I have fish **and** beer with the same meal, I **abstain** from ice cream. **When** I eat ice cream **or don't** drink beer, **then** I **never** touch fish.

- ▶ Every sentence is a proposition that consists of sub-propositions (e. g., “eat ice cream or don't drink beer”).
- ▶ atomic propositions “drink beer”, “eat fish”, “eat ice cream”
- ▶ logical connectives “**and**”, “**or**”, **negation**, “**if**, **then**”

Exercise from U. Schöning: Logik für Informatiker
Picture courtesy of graur razvan ionut / FreeDigitalPhotos.net

Problems with Natural Language



If I don't drink beer **to a meal**, then I **always** eat fish.

Whenever I have fish and beer **with the same meal**, I abstain from ice cream.

When I eat ice cream or don't drink beer, then I never touch fish.

► “irrelevant” information

Exercise from U. Schöning: Logik für Informatiker
Picture courtesy of graur razvan ionut / FreeDigitalPhotos.net

Problems with Natural Language



If I **don't** drink beer, then I eat fish.
Whenever I have fish and beer, I **abstain**
from ice cream.
When I eat ice cream or **don't** drink
beer, then I **never** touch fish.

- ▶ “irrelevant” information
- ▶ **different formulations for the same connective/proposition**

Exercise from U. Schöning: Logik für Informatiker
Picture courtesy of graur razvan ionut / FreeDigitalPhotos.net

Problems with Natural Language



If not DrinkBeer, then EatFish.
If EatFish and DrinkBeer,
then not EatIceCream.
If EatIceCream or not DrinkBeer,
then not EatFish.

- ▶ “irrelevant” information
- ▶ different formulations for the same connective/proposition

Exercise from U. Schöning: Logik für Informatiker
Picture courtesy of graur razvan ionut / FreeDigitalPhotos.net

What is Next?

- ▶ What are meaningful (well-defined) sequences of atomic propositions and connectives?
“if then EatIceCream not or DrinkBeer and” not meaningful
→ **syntax**
- ▶ What does it mean if we say that a statement is true?
Is “DrinkBeer and EatFish” true?
→ **semantics**
- ▶ When does a statement logically follow from another?
Does “EatFish” follow from “if DrinkBeer, then EatFish”?
→ **logical entailment**

German: Syntax, Semantik, logische Folgerung

E1.2 Syntax of Propositional Logic

Syntax of Propositional Logic

Definition (Syntax of Propositional Logic)

Let A be a set of **atomic propositions**. The set of **propositional formulas** (over A) is inductively defined as follows:

- ▶ Every **atom** $a \in A$ is a propositional formula over A .
- ▶ If φ is a propositional formula over A , then so is its **negation** $\neg\varphi$.
- ▶ If φ and ψ are propositional formulas over A , then so is the **conjunction** $(\varphi \wedge \psi)$.
- ▶ If φ and ψ are propositional formulas over A , then so is the **disjunction** $(\varphi \vee \psi)$.

The **implication** $(\varphi \rightarrow \psi)$ is an abbreviation for $(\neg\varphi \vee \psi)$.

The **biconditional** $(\varphi \leftrightarrow \psi)$ is an abbrev. for $((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi))$.

German: atomare Aussage, aussagenlogische Formel, Atom, Negation, Konjunktion, Disjunktion, Implikation, Bikonditional

Syntax: Examples

Which of the following sequences of symbols are propositional formulas over the set of all possible letter sequences? Which kinds of formula are they (atom, conjunction, ...)?

- ▶ $(A \wedge (B \vee C))$
- ▶ $((\text{EatFish} \wedge \text{DrinkBeer}) \rightarrow \neg \text{EatIceCream})$
- ▶ $\neg(\wedge \text{Rain} \vee \text{StreetWet})$
- ▶ $\neg(\text{Rain} \vee \text{StreetWet})$
- ▶ $\neg(A = B)$
- ▶ $(A \wedge \neg(B \leftrightarrow)C)$
- ▶ $(A \vee \neg(B \leftrightarrow C))$
- ▶ $((A \leq B) \wedge C)$
- ▶ $((A_1 \wedge A_2) \vee \neg(A_3 \leftrightarrow A_2))$

E1.3 Semantics of Propositional Logic

Meaning of Propositional Formulas?

So far propositional formulas are only symbol sequences without any meaning.

For example, what does this mean:

$((\text{EatFish} \wedge \text{DrinkBeer}) \rightarrow \neg \text{EatIceCream})?$

▷ **We need semantics!**

Semantics of Propositional Logic

Definition (Semantics of Propositional Logic)

A **truth assignment** (or **interpretation**) for a set of atomic propositions A is a function $\mathcal{I} : A \rightarrow \{0, 1\}$.

A propositional **formula** φ (over A) **holds under** \mathcal{I} (written as $\mathcal{I} \models \varphi$) according to the following definition:

$\mathcal{I} \models a$	iff	$\mathcal{I}(a) = 1$	(for $a \in A$)
$\mathcal{I} \models \neg\varphi$	iff	not $\mathcal{I} \models \varphi$	
$\mathcal{I} \models (\varphi \wedge \psi)$	iff	$\mathcal{I} \models \varphi$ and $\mathcal{I} \models \psi$	
$\mathcal{I} \models (\varphi \vee \psi)$	iff	$\mathcal{I} \models \varphi$ or $\mathcal{I} \models \psi$	

Question: should we define semantics of $(\varphi \rightarrow \psi)$ and $(\varphi \leftrightarrow \psi)$?

German: Wahrheitsbelegung/Interpretation, φ gilt unter \mathcal{I}

Semantics of Propositional Logic: Terminology

- ▶ For $\mathcal{I} \models \varphi$ we also say \mathcal{I} is a model of φ and that φ is true under \mathcal{I} .
- ▶ If φ does not hold under \mathcal{I} , we write this as $\mathcal{I} \not\models \varphi$ and say that \mathcal{I} is no model of φ and that φ is false under \mathcal{I} .
- ▶ **Note:** \models is not part of the formula but part of the meta language (speaking about a formula).

German: \mathcal{I} ist ein/kein Modell von φ ; φ ist wahr/falsch unter \mathcal{I} ;
Metasprache

Exercise

Consider set $A = \{X, Y, Z\}$ of atomic propositions and formula $\varphi = (X \wedge \neg Y)$.

Specify an interpretation \mathcal{I} for A with $\mathcal{I} \models \varphi$.

Semantics: Example (1)

$$A = \{\text{DrinkBeer}, \text{EatFish}, \text{EatIceCream}\}$$

$$\mathcal{I} = \{\text{DrinkBeer} \mapsto 1, \text{EatFish} \mapsto 0, \text{EatIceCream} \mapsto 1\}$$

$$\varphi = (\neg\text{DrinkBeer} \rightarrow \text{EatFish})$$

Do we have $\mathcal{I} \models \varphi$?

Semantics: Example (2)

Goal: prove $\mathcal{I} \models \varphi$.

Let us use the definitions we have seen:

$$\begin{aligned}\mathcal{I} \models \varphi &\text{ iff } \mathcal{I} \models (\neg\text{DrinkBeer} \rightarrow \text{EatFish}) \\ &\text{ iff } \mathcal{I} \models (\neg\neg\text{DrinkBeer} \vee \text{EatFish}) \\ &\text{ iff } \mathcal{I} \models \neg\neg\text{DrinkBeer} \text{ or } \mathcal{I} \models \text{EatFish}\end{aligned}$$

This means that if we want to prove $\mathcal{I} \models \varphi$, it is sufficient to prove

$$\mathcal{I} \models \neg\neg\text{DrinkBeer}$$

or to prove

$$\mathcal{I} \models \text{EatFish}.$$

We attempt to prove the first of these statements.

Semantics: Example (3)

New goal: prove $\mathcal{I} \models \neg\neg\text{DrinkBeer}$.

We again use the definitions:

$$\begin{aligned}\mathcal{I} \models \neg\neg\text{DrinkBeer} &\text{ iff not } \mathcal{I} \models \neg\text{DrinkBeer} \\ &\text{ iff not not } \mathcal{I} \models \text{DrinkBeer} \\ &\text{ iff } \mathcal{I} \models \text{DrinkBeer} \\ &\text{ iff } \mathcal{I}(\text{DrinkBeer}) = 1\end{aligned}$$

The last statement is true for our interpretation \mathcal{I} .

To write this up as a **proof** of $\mathcal{I} \models \varphi$,
we can go through this line of reasoning back-to-front,
starting from our assumptions and ending with the conclusion
we want to show.

Semantics: Example (4)

Let $\mathcal{I} = \{\text{DrinkBeer} \mapsto 1, \text{EatFish} \mapsto 0, \text{EatIceCream} \mapsto 1\}$.

Proof that $\mathcal{I} \models (\neg\text{DrinkBeer} \rightarrow \text{EatFish})$:

- ① We have $\mathcal{I} \models \text{DrinkBeer}$
(uses defn. of \models for atomic props. and fact $\mathcal{I}(\text{DrinkBeer}) = 1$).
- ② From (1), we get $\mathcal{I} \not\models \neg\text{DrinkBeer}$
(uses defn. of \models for negations).
- ③ From (2), we get $\mathcal{I} \models \neg\neg\text{DrinkBeer}$
(uses defn. of \models for negations).
- ④ From (3), we get $\mathcal{I} \models (\neg\neg\text{DrinkBeer} \vee \psi)$ for all formulas ψ ,
in particular $\mathcal{I} \models (\neg\neg\text{DrinkBeer} \vee \text{EatFish})$
(uses defn. of \models for disjunctions).
- ⑤ From (4), we get $\mathcal{I} \models (\neg\text{DrinkBeer} \rightarrow \text{EatFish})$
(uses defn. of “ \rightarrow ”).



Summary

- ▶ **propositional logic** based on atomic propositions
- ▶ **syntax** defines what well-formed formulas are
- ▶ **semantics** defines when a formula is true
- ▶ **interpretations** are the basis of semantics