

# Discrete Mathematics in Computer Science

## D1. Introduction to Recurrences

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## — D1. Introduction to Recurrences

### D1.1 Recurrences

### D1.2 Examples of Recurrences

### D1.3 Fibonacci Series – Mathematical Induction

## D1.1 Recurrences

### Recursion (1)

The concept of **recursion** is very common in computer science and discrete mathematics.

- ▶ When **designing algorithms**, recursion relates to the idea of solving a problem by solving smaller subproblems of the same kind.
- ▶ **Examples:**
  - ▶ For example, we can sort a sequence by sorting smaller subsequences and then combining the result  $\rightsquigarrow$  **mergesort**
  - ▶ We can find an element in a sorted sequence by identifying which half of the sequence the element must be located in, and then searching this half  $\rightsquigarrow$  **binary search**
  - ▶ We can insert elements into a search tree by identifying which child of the root node the element must be added to, then recursively inserting it there  $\rightsquigarrow$  **trees as data structures**

## Recursion (2)

The concept of **recursion** is very common in computer science and discrete mathematics.

- ▶ When **designing data structures**, it is often helpful to think of a data structures as being composed of smaller data structures of the same kind.
- ▶ **Examples:**
  - ▶ A **rooted binary tree** is either a leaf or an inner node with two children, which are themselves rooted binary trees.
  - ▶ A **singly linked list** is either empty or a head element followed by a tail, which is itself a linked list.
  - ▶ A **logical formula** is either an atomic formula or a composite formula, which consists of one of two formulas connected by logical connectives ("and", "or", "not").

## Recurrences

In this part of the lecture, we study **recurrences**, i.e., recursively defined functions  $f : \mathbb{N}_0 \rightarrow \mathbb{R}$  where  $f(n)$  is defined in terms of the values  $f(m)$  for  $m < n$ .

- ▶ Such recurrences naturally arise in all mentioned applications.
- ▶ They are particularly useful for studying the runtime of algorithms, especially recursive algorithms.

## Recursion (3)

The concept of **recursion** is very common in computer science and discrete mathematics.

- ▶ In **combinatorial counting problems**, counting things often involves solving smaller counting problems of the same type and combining the results.
- ▶ **Examples:**
  - ▶ counting the number of **subsets of size  $k$**  of a set of size  $n$
  - ▶ counting the number of **permutations** of a set of size  $n$
  - ▶ counting the number of **rooted binary trees** with  $n$  leaves

## Learning Objectives

- ▶ Recurrences are a wide topic, and in our brief coverage we will only scratch the surface.
- ▶ Our aim is to equip you with enough knowledge to
  - ▶ understand **what recurrences are**
  - ▶ understand **where they arise**
  - ▶ understand **why they are of interest**
  - ▶ get to know some **important examples** of recurrences, such as the **Fibonacci series**
  - ▶ get a feeling for some **mathematical techniques** used to solve recurrences, in particular:
    - ▶ **mathematical induction**
    - ▶ **generating functions**
    - ▶ the **master theorem** for divide-and-conquer recurrences
  - ▶ **apply the master theorem in practice**

## D1.2 Examples of Recurrences

### Counting Permutations

Let  $S$  be a finite set, and let  $n = |S|$ .

Question: How many permutations of  $S$  exist?

We answer this question by answering the following slightly more general question:

Let  $X$  and  $Y$  be finite sets, and let  $n = |X| = |Y|$ .

Question: How many bijective functions from  $X$  to  $Y$  exist?

The permutation question is the special case where  $S = X = Y$ .

## Examples of Recurrences

In this section, we look at three recurrences that arise in **combinatorics**, i.e., when **counting** things:

- ▶ **factorials**: counting permutations
- ▶ **binomial coefficients**: counting subsets of a certain size
- ▶ **Catalan numbers**: counting rooted binary trees

We also have a first look at the **Fibonacci series**, perhaps the most famous recurrence in mathematics.

### Counting Permutations

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Question: How many bijective functions from  $X$  to  $Y$  exist?

The permutation question is the special case where  $S = X = Y$ .

### Counting Bijections – Derivation

How many bijective functions from  $X$  to  $Y$  exist, where  $n = |X| = |Y|$ ?

Denote this number by  $f(n)$ .

- ▶ We have  $f(0) = 1$ : there exists one possible function from  $X = \emptyset$  to  $Y = \emptyset$  (the empty function), and it is bijective.
- ▶ For  $n \geq 1$ , let  $x \in X$  be any element of  $X$ .
  - ▶ Every bijection  $g : X \rightarrow Y$  maps  $x$  to some element  $g(x) = y \in Y$ .
  - ▶ There are  $n = |Y|$  possible choices for  $y$ .
- ▶ In order to be bijective,  $g$  must bijectively map all **other** elements in  $X$  to **other** elements of  $Y$ .
  - ▶ Hence,  $g$  restricted to  $X \setminus \{x\}$  is a bijective function from  $X \setminus \{x\}$  to  $Y \setminus \{y\}$ .
  - ▶ Because  $|X \setminus \{x\}| = |Y \setminus \{y\}| = n - 1$ , there are  $f(n - 1)$  choices for these mappings.
- ▶ This gives us  $f(n) = n \cdot f(n - 1)$  for all  $n \geq 1$ .

## Counting Bijections – Result

### Theorem

The number of bijections between finite sets of size  $n$ , or equivalently the number of permutations of a finite set of size  $n$ , is given by the recurrence:

$$\begin{aligned} f(0) &= 1 \\ f(n) &= n \cdot f(n-1) \quad \text{for all } n \geq 1 \end{aligned}$$

Closed-form solution:

$$f(n) = n!$$

## Counting $k$ -Subsets

Let  $S$  be a finite set, let  $n = |S|$ , and let  $k \in \{0, \dots, n\}$ .

Question: How many subsets of  $S$  of size  $k$  exist?

Denote this number by  $\binom{n}{k}$ .

- ▶ We have  $\binom{n}{0} = 1$ : the only subset of size 0 is  $\emptyset$ .
- ▶ We have  $\binom{n}{n} = 1$ : the only subset of size  $n$  is  $S$  itself.
- ▶ For all other cases, we count proper, nontrivial subsets. Let  $x \in S$  be any element.
- ▶ There are two kinds of subsets of  $S$  of size  $k$ :
  - ▶ subsets that do not include  $x$ : Such subsets include  $k$  elements of the set  $S \setminus \{x\}$ . Because  $|S \setminus \{x\}| = n-1$ , there are  $\binom{n-1}{k}$  such subsets.
  - ▶ subsets that include  $x$ : Such subsets include  $k-1$  elements of  $S \setminus \{x\}$ . Because  $|S \setminus \{x\}| = n-1$ , there are  $\binom{n-1}{k-1}$  such subsets.
- ▶ In summary:  $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$  for all  $n \geq 1$  and  $0 < k < n$ .

## Counting $k$ -Subsets – Result

### Theorem

Let  $S$  be a finite set with  $n$  elements, and let  $k \in \{0, \dots, n\}$ . Then  $S$  has  $\binom{n}{k}$  subsets of size  $k$ , where

$$\begin{aligned} \binom{n}{0} &= 1 \\ \binom{n}{n} &= 1 \\ \binom{n}{k} &= \binom{n-1}{k} + \binom{n-1}{k-1} \quad \text{for all } n \geq 1, 0 < k < n \end{aligned}$$

Closed-form solution:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

## Counting $k$ -Subsets – Proof of Closed-Form Solution

To prove that the given closed-form solution is correct, it suffices to verify that it satisfies the recurrence:

- ▶ case  $k = 0$ :  $\frac{n!}{k!(n-k)!} = \frac{n!}{0!(n-0)!} = \frac{n!}{1 \cdot n!} = 1 = \binom{n}{0}$ .
- ▶ case  $k = n$ :  $\frac{n!}{k!(n-k)!} = \frac{n!}{n!(n-n)!} = \frac{n!}{n! \cdot 0!} = \frac{n!}{n! \cdot 1} = 1 = \binom{n}{n}$ .
- ▶ case  $0 < k < n$ :

$$\begin{aligned} & \frac{(n-1)!}{k!((n-1)-k)!} + \frac{(n-1)!}{(k-1)!((n-1)-(k-1))!} \\ &= \frac{(n-1)!}{k!(n-k-1)!} + \frac{(n-1)!}{(k-1)!(n-k)!} \\ &= \frac{(n-1)! \cdot (n-k)}{k!(n-k-1)! \cdot (n-k)} + \frac{(n-1)! \cdot k}{(k-1)! \cdot k \cdot (n-k)!} \\ &= \frac{(n-1)! \cdot (n-k)}{k!(n-k)!} + \frac{(n-1)! \cdot k}{k! \cdot (n-k)!} \\ &= \frac{(n-1)! \cdot ((n-k)+k)}{k!(n-k)!} = \frac{(n-1)! \cdot n}{k!(n-k)!} = \frac{n!}{k!(n-k)!} \end{aligned}$$

## Binary Trees

### Definition (binary tree)

A **binary tree** is inductively defined as a tuple of the following form:

- ▶ The **empty tree** () is a binary tree.  
Such a tree is called a **leaf**.
- ▶ If  $L$  and  $R$  are binary trees, then  $(L, R)$  is a binary tree.  
Such a tree is called an **inner node**  
with **left child**  $L$  and **right child**  $R$ .

**German:** Binärbaum

**Note:** With these kinds of trees, the order of children matters, i.e.,  $(L, R)$  and  $(R, L)$  are different trees (unless  $L = R$ ).

## Counting Binary Trees

**Question:** How many binary trees with  $n + 1$  leaves exist?  
(Why  $n + 1$ ?)

Denote this number by  $C(n)$ .

- ▶ We have  $C(0) = 1$ : () is the only tree with one leaf.
- ▶ For  $n \geq 1$ , the tree must be an inner node.  
Each child must have between 1 and  $n$  leaves.  
The number of leaves of the children must sum to  $n + 1$ .
- ▶ Hence, if the left child has  $k + 1$  leaves, the right child has  $(n + 1) - (k + 1) = n - k = (n - k - 1) + 1$  leaves.
- ▶ We obtain:  $C(n) = \sum_{k=0}^{n-1} C(k)C(n - k - 1)$ .

## Counting Binary Trees – Result

### Theorem

There are  $C(n)$  binary trees with  $n + 1$  leaves, where

$$C(0) = 1$$

$$C(n) = \sum_{k=0}^{n-1} C(k)C(n - k - 1) \quad \text{for all } n \geq 1$$

**Closed-form solution (without proof):**

$$C(n) = \frac{1}{n+1} \binom{2n}{n}$$

## Catalan Numbers

The numbers  $C(n)$  are called **Catalan numbers**  
after 19th century Belgian mathematician Eugène Charles Catalan.

First terms of the Catalan sequence:

1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, ...

## Fibonacci Series

- ▶ The last recurrence we consider in this section is the famous **Fibonacci series** (or Fibonacci sequence).
- ▶ We directly introduce it with its definition as a recurrence rather than via an application.

## Fibonacci Series – Definition

### Definition (Fibonacci series)

The **Fibonacci series**  $F$  is defined as follows:

$$F(0) = 0$$

$$F(1) = 1$$

$$F(n) = F(n - 1) + F(n - 2) \quad \text{for all } n \geq 2$$

**German:** Fibonacci-Folge

First terms of the Fibonacci series:

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ...

**Closed-form solution:**  $\rightsquigarrow$  next section

## Fibonacci Series – Trivia

- ▶ The Fibonacci series is named after Leonardo of Pisa a.k.a. **Fibonacci** (son of Bonacci), who introduced it to Western Europe in the 13th century.
- ▶ It has been known outside Europe much earlier, dating back to the Indian mathematician Pingala (3rd century BCE).
- ▶ The series has many, many applications.
- ▶ There exist mathematical journals solely dedicated to it, the most famous one being “Fibonacci Quarterly”.

## D1.3 Fibonacci Series – Mathematical Induction

## Overview

- ▶ In this section, we prove a closed-form expression for the Fibonacci series.
- ▶ We do this because the result itself is interesting (because of the many applications of the Fibonacci series), but also to practice proving closed-form expressions for recurrences by mathematical induction.
- ▶ In the next section, we describe a more advanced technique with which we cannot just **prove** the given expression but also **derive** it ourselves.

## Negative Inverse of the Golden Ratio

### Definition (negative inverse of the golden ratio)

The

$$\psi = \frac{1 - \sqrt{5}}{2}$$

is called the **negative inverse of the golden ratio**.

- ▶ Numerically,  $\psi = -0.618034$  (approximately).
- ▶ The name for  $\psi$  derives from the fact that  $\psi = -\frac{1}{\varphi}$ . However, we do not need this property here, and therefore we do not prove it.

## Golden Ratio

### Definition (golden ratio)

The number

$$\varphi = \frac{1 + \sqrt{5}}{2}$$

is called the **golden ratio**.

**German:** goldener Schnitt

- ▶ Numerically,  $\varphi = 1.618034$  (approximately).
- ▶ The golden ratio is a famous mathematical constant because it naturally occurs in many contexts and because of its aesthetical properties.

## Fibonacci Series – Closed-Form Expression

### Theorem

$$\begin{aligned} F(n) &= \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right) \\ &= \frac{1}{\sqrt{5}} (\varphi^n - \psi^n) \end{aligned} \quad \text{for all } n \geq 0$$

Before we prove the theorem, we prove a number of lemmas.

- ▶ Note that  $|\psi| < 1$  and hence  $\psi^n \rightarrow 0$  as  $n \rightarrow \infty$ .
- ▶ With some calculation, we see that we can alternatively compute  $F(n)$  by rounding  $\frac{1}{\sqrt{5}} \varphi^n$  to the nearest integer, ignoring the  $\psi^n$  term.

## First Lemma

### Lemma

$$\psi = 1 - \varphi$$

### Proof.

$$\begin{aligned}\psi &= \frac{1 - \sqrt{5}}{2} \\ &= \frac{1 + 1 - 1 - \sqrt{5}}{2} \\ &= \frac{2 - (1 + \sqrt{5})}{2} \\ &= \frac{2}{2} - \frac{1 + \sqrt{5}}{2} \\ &= 1 - \varphi\end{aligned}$$

□

## Second Lemma

### Lemma

$$\varphi^2 = \varphi + 1$$

### Proof.

$$\begin{aligned}\varphi^2 &= \left(\frac{1 + \sqrt{5}}{2}\right)^2 = \frac{1}{4}(1 + \sqrt{5})^2 \\ &= \frac{1}{4}(1 + 2\sqrt{5} + 5) \\ &= \frac{1}{4}(2 + 2\sqrt{5} + 4) = \frac{1}{4}(2 + 2\sqrt{5}) + \frac{4}{4} \\ &= \frac{1}{2}(1 + \sqrt{5}) + 1 \\ &= \varphi + 1\end{aligned}$$

□

## Third Lemma

### Lemma

$$\psi^2 = \psi + 1$$

### Proof.

$$\begin{aligned}\psi^2 &= (1 - \varphi)^2 \\ &= 1 - 2\varphi + \varphi^2 \\ &= 1 - 2\varphi + \varphi + 1 \\ &= 1 - \varphi + 1 \\ &= (1 - \varphi) + 1 \\ &= \psi + 1\end{aligned}$$

□

## Main Proof (1)

### Reminders:

$$\begin{aligned}F(0) &= 0 & F(1) &= 1 & F(n) &= F(n-1) + F(n-2) \text{ for all } n \geq 2 \\ \varphi^2 &= \varphi + 1 & \psi^2 &= \psi + 1 & \text{Claim: } F(n) &= \frac{1}{\sqrt{5}}(\varphi^n - \psi^n)\end{aligned}$$

### Proof.

Proof by (strong) induction over  $n$ .

First base case  $n = 0$ :

$$\frac{1}{\sqrt{5}}(\varphi^0 - \psi^0) = \frac{1}{\sqrt{5}}(1 - 1) = 0 = F(0)$$

Second base case  $n = 1$ :

$$\begin{aligned}\frac{1}{\sqrt{5}}(\varphi^1 - \psi^1) &= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}-1+\sqrt{5}}{2}\right) \\ &= \frac{1}{\sqrt{5}}\left(\frac{2\sqrt{5}}{2}\right) = 1 = F(1)\end{aligned}$$

...

## Main Proof (2)

Reminders:

$$\begin{aligned} F(0) &= 0 & F(1) &= 1 & F(n) &= F(n-1) + F(n-2) \text{ for all } n \geq 2 \\ \varphi^2 &= \varphi + 1 & \psi^2 &= \psi + 1 & \text{Claim: } F(n) &= \frac{1}{\sqrt{5}}(\varphi^n - \psi^n) \end{aligned}$$

Proof (continued).

Induction step ( $n$  building on  $n-1$  and  $n-2$ ):

$$\begin{aligned} F(n) &= F(n-1) + F(n-2) \\ &= \frac{1}{\sqrt{5}}(\varphi^{n-1} - \psi^{n-1}) + \frac{1}{\sqrt{5}}(\varphi^{n-2} - \psi^{n-2}) \\ &= \frac{1}{\sqrt{5}}(\varphi^{n-1} + \varphi^{n-2} - (\psi^{n-1} + \psi^{n-2})) \\ &= \frac{1}{\sqrt{5}}(\varphi^{n-2}(\varphi + 1) - \psi^{n-2}(\psi + 1)) \\ &= \frac{1}{\sqrt{5}}(\varphi^{n-2} \cdot \varphi^2 - \psi^{n-2} \cdot \psi^2) = \frac{1}{\sqrt{5}}(\varphi^n - \psi^n) \end{aligned}$$

□