Discrete Mathematics in Computer Science

B10. A Glimpse of Abstract Algebra

Malte Helmert, Gabriele Röger

University of Basel

B10.1 Abstract Groups

Abstract Algebra

- Elementary algebra: "Arithmetics with variables"
  - e. g. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ describes the solutions of $ax^2 + bx + c = 0$ where $a \neq 0$.
  - Variables for numbers and operations such as addition, subtraction, multiplication, division ...
  - "What you learn at school."
- Abstract algebra: Generalization of elementary algebra
  - Arbitrary sets and operations on their elements
  - e. g. permutations of a given set S plus function composition
  - Abstract algebra studies arbitrary sets and operations based on certain properties (such as associativity).

Binary operations

- A binary operation on a set S is a function $f : S \times S \to S$.
- e. g. $add : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$ for addition of natural numbers.
- In infix notation, we write the operator between the operands, e. g. $x + y$ instead of $add(x, y)$.

Groups

Definition (Group)

A group $G = (S, \cdot)$ is given by a set S and a binary operation $\cdot$ on S that satisfy the group axioms:

- Associativity: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in S$.
- Identity element: There exists an $e \in S$ such that for all $x \in S$ it holds that $x \cdot e = e \cdot x = x$.
  Element e is called identity or neutral element of the group.
- Inverse element: For every $x \in S$ there is a $y \in S$ such that $x \cdot y = y \cdot x = e$, where e is the identity element.

A group is called abelian if $\cdot$ is also commutative, i. e. for all $x, y \in S$ it holds that $x \cdot y = y \cdot x$.

Cardinality $|S|$ is called the order of the group.

Niels Henrik Abel: Norwegian mathematician (1802–1829), cf. Abel prize

Example: $(\mathbb{Z}, +)$

$(\mathbb{Z}, +)$ is a group:

- $\mathbb{Z}$ is closed under addition, i. e. for $x, y \in \mathbb{Z}$ it holds that $x + y \in \mathbb{Z}$
- The $+$ operator is associative: for all $x, x, z \in \mathbb{Z}$ it holds that $(x + y) + z = x + (y + z)$.
- Integer 0 is the neutral element: for all integers x it holds that $x + 0 = 0 + x = x$.
- Every integer x has an inverse element in the integers, namely $-x$, because $x + (-x) = (-x) + x = 0$.

$(\mathbb{Z}, +)$ also is an abelian group
because for all $x, y \in \mathbb{Z}$ it holds that $x + y = y + x$.

Uniqueness of Identity and Inverses

Theorem

Every group $G = (S, \cdot)$ has only one identity element and for each $x \in S$ the inverse of x is unique.

Proof.

identity: Assume that there are two identity elements $e, e' \in S$ with $e \neq e'$. Then for all $x \in S$ it holds that $x \cdot e = e \cdot x = x$ and that $x \cdot e' = e' \cdot x = x$. Using $x = e'$, we get $e' \cdot e = e'$ and using $x = e$ we get $e' \cdot e = e$, so overall $e' = e$. ↯

inverse: homework assignment □

We often denote the identity element with 1 and the inverse of x with x^{-1}.

Division – Right Quotient

Theorem

Let $G = (S, \cdot)$ be a group. Then for all $a, b \in S$ the equation $x \cdot b = a$ has exactly one solution x in S, namely $x = a \cdot b^{-1}$.

We call $a \cdot b^{-1}$ the right-quotient of a by b and also write it as a/b.

Proof.

It is a solution: With $x = a \cdot b^{-1}$ it holds that
$x \cdot b = (a \cdot b^{-1}) \cdot b = a \cdot (b^{-1} \cdot b) = a \cdot 1 = a$.

The solution is unique:
Assume x and x' are distinct solutions. Then $x \cdot b = a = x' \cdot b$.
Multiplying both sides by b^{-1}, we get $(x \cdot b) \cdot b^{-1} = (x' \cdot b) \cdot b^{-1}$ and with associativity $x \cdot (b \cdot b^{-1}) = x' \cdot (b \cdot b^{-1})$.
With the axiom on inverse elements this leads to $x \cdot 1 = x' \cdot 1$ and with the axiom on the identity element ultimately to $x = x'$. $\lightning$ □

Division – Left Quotient

Theorem

Let $G = (S, \cdot)$ be a group. Then for all $a, b \in S$ the equation $b \cdot x = a$ has exactly one solution x in S, namely $x = b^{-1} \cdot a$.

We call $b^{-1} \cdot a$ the left-quotient of a by b and also write it as $b \setminus a$.

Proof omitted

Quotients in Abelian Groups

Theorem

If $G = (S, \cdot)$ is an abelian group then it holds for all $x, y \in S$ that $x/y = y\backslash x$.

Proof.

Consider arbitrary $x, y \in S$. As $\cdot$ is commutative, it holds that $x/y = x \cdot y^{-1} = y^{-1} \cdot x = y\backslash x$. □

Group Homomorphism

A group homomorphism is a function that preserves group structure:

Definition (Group homomorphism)

Let $G = (S, \cdot)$ and $G' = (S', \circ)$ be groups.
A homomorphism from G to G' is a function $f : S \to S'$ such that for all $x, y \in S$ it holds that $f(x \cdot y) = f(x) \circ f(y)$.

Definition (Group Isomorphism)

A group homomorphism that is bijective is called a group isomorophism. Groups G and H are called isomorphic if there is a group isomorphism from G to H.

From a practical perspective, isomorphic groups are identical up to renaming.

Group Homomorphism – Example

- Consider $G = (\mathbb{Z}, +)$ and $H = (\{1, -1\}, \cdot)$ with
  - $1 \cdot 1 = -1 \cdot -1 = 1$
  - $1 \cdot -1 = -1 \cdot 1 = -1$
- Let $f : \mathbb{Z} \to \{1, -1\}$ with $f(x) = \begin{cases} 1 & \text{if } x \text{ is even} \\ -1 & \text{if } x \text{ is odd} \end{cases}$
- f is a homomorphism from G to H:
  for all $x, y \in \mathbb{Z}$ it holds that

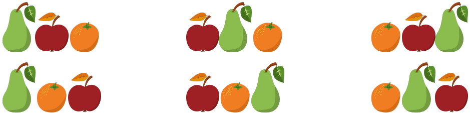
$$
\begin{aligned}
f(x+y) &= \begin{cases} 1 & \text{if } x+y \text{ is even} \\ -1 & \text{if } x+y \text{ is odd} \end{cases} \\
&= \begin{cases} 1 & \text{if } x \text{ and } y \text{ have the same parity} \\ -1 & \text{if } x \text{ and } y \text{ have different parity} \end{cases} \\
&= \begin{cases} 1 & \text{if } f(x) = f(y) \\ -1 & \text{if } f(x) \neq f(y) \end{cases} \\
&= f(x) \cdot f(y)
\end{aligned}
$$

Outlook

- A subgroup of $G = (S, \cdot)$ is a group $H = (S', \circ)$ with $S' \subseteq S$ and $\circ$ the restriction of $\cdot$ to $S' \times S'$.
  - S' always contains the identity element and is closed under group operation and inverse
  - group homomorphisms preserve many properties of subgroups
- Other algebraic structures, e. g.
  - Semi-group: requires only associativity
  - Monoid: requires associativity and identity element
  - Ringoids: algebraic structures with two binary operations
    - multiplication and addition
    - multiplication distributes over addition
    - e. g. ring and field

B10.2 Symmetric Group and Permutation Groups

Reminder: Permutations



Definition (Permutation)

Let S be a set. A bijection $\pi : S \to S$ is called a permutation of S.

Symmetric Group

Theorem (Symmetric Group)

Let M be a set. Then $\mathrm{Sym}(M) = (S, \cdot)$, *where*

- *S is the set of all permutations of M, and*
- *$\cdot$ denotes function composition,*

is a group, called the symmetric group of M.

For finite set $M = \{1, \dots, n\}$, we also use S_n to refer to the symmetric group of M.

Is the symmetric group abelian?

What's the order of S_n?

Symmetric Group – Proof I

Theorem

For set M, $\mathrm{Sym}(M) = (\{\sigma : M \to M \mid \sigma \text{ is bijective}\}, \cdot)$ is a group.

Definition (Group)

A group $G = (S, \cdot)$ is given by a set S and
a binary operation $\cdot$ on S that satisfy the group axioms:

- Associativity: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in S$.
- Identity element: There exists an $e \in S$ such that for all $x \in S$ it holds that $x \cdot e = e \cdot x = x$. Element e is called identity of neutral element of the group.
- Inverse element: For every $x \in S$ there is a $y \in S$ such that $x \cdot y = y \cdot x = e$, where e is the identity element.

To show: closure, associativity, identity, inverse element

Symmetric Group – Proof II

Theorem

For set M, $\mathrm{Sym}(M) = (\{\sigma : M \to M \mid \sigma \text{ is bijective}\}, \cdot)$ is a group.

Proof.

- Closure: The product of two permutations of M is a permutation of M and hence in the set.
- Associativity: Function composition is always associative.
- Identity element: Function $\mathrm{id} : M \to M$ with $\mathrm{id}(x) = x$ is a permutation and for every permutation σ of M it holds that $\sigma\mathrm{id} = \mathrm{id}\sigma = \sigma$.
- Inverse element: For every permutation σ of M, also the inverse function σ^{-1} is a permutation of M and has the required properties.

□

Generating Sets

Definition

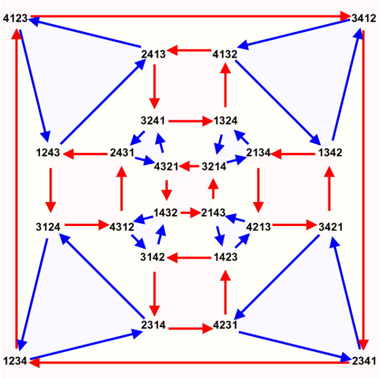
A generating set of a group $G = (S, \circ)$ is a set $S' \subseteq S$ such that every $e \in S$ can be expressed as a combination (under $\circ$) of finitely many elements of S' and their inverses.

Empty product is identity by definition, so no need to have it in S'.

- For $n \geq 2$, S_n is generated by $\{(i \quad i+1) \mid i \in \{1, \dots, n-1\}\}$.
- For $n > 2$, S_n is generated by $\{(1 \quad 2), (1 \quad \dots \quad n)\}$.

Generating Sets – Example

$$\left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} \right\} \text{ is a generating set of } S_4.$$



Permutation Group

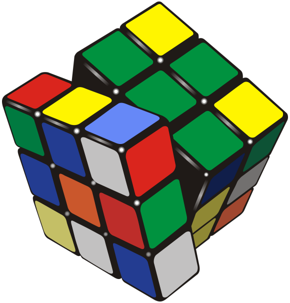
Sometimes, we do not want to consider all possible permutations.

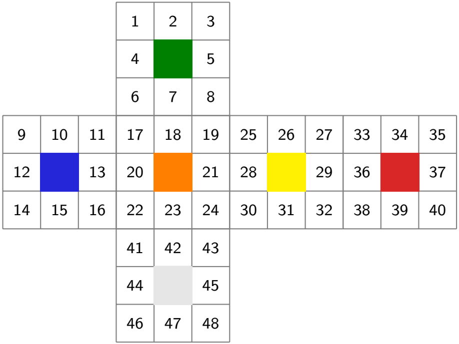
Definition (Permutation Group)

A permutation group is a group $G = (S, \cdot)$,
where S is a set of permutations of some set M and
$\cdot$ is the composition of permutations in S.

Every permutation group is a subgroup of a symmetric group and every such subgroup is a permutation group.

Permutation Group – Example





- Consider all permutations achievable with valid moves.
- Subgroup of S_{48} with order $43\,252\,003\,274\,489\,856\,000 \approx 4.3 \cdot 10^{19}$ (43 quintillion)
- S_{48} has order $48! \approx 1.24 \cdot 10^{61}$