Discrete Mathematics in Computer Science

B9. Permutations

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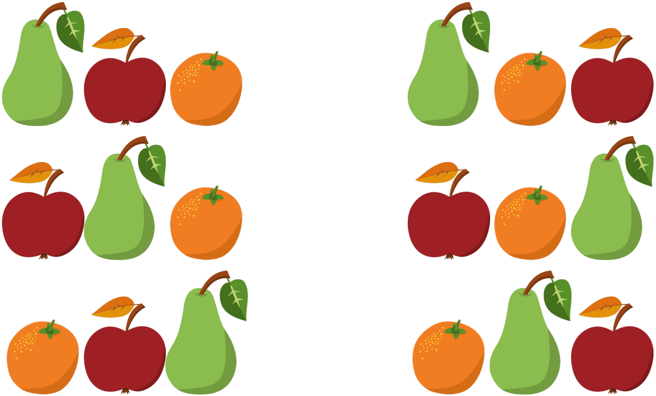


Permutations as Functions

- A permutation rearranges objects.
- Consider for example sequence o_2, o_1, o_3, o_4
- Let's rearrange the objects, e. g. to o_3, o_1, o_4, o_2.
  - The object at position 1 was moved to position 4,
  - the one from position 3 to position 1,
  - the one from position 4 to position 3 and
  - the one at position 2 stayed where it was.
- This corresponds to a bijection $\sigma : \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$ with $\sigma(1) = 4$, $\sigma(2) = 2$, $\sigma(3) = 1$, $\sigma(4) = 3$
- We call such a bijection a permutation.

Permutation – Definition

Definition (Permutation)

Let S be a set. A bijection $\pi : S \to S$ is called a permutation of S.

We will focus on permutations of finite sets.

The actual objects in S don't matter,
so we mostly work with $\{1, \dots, |S|\}$.

How many permutations are there for a finite set S?

Two-line and One-line Notation (for Finite Sets)

Consider π with
$\pi(1) = 2, \pi(2) = 5, \pi(3) = 4, \pi(4) = 3, \pi(5) = 1, \pi(6) = 6.$

Two-line notation lists the elements of S in the first row and the image of each element in the second row:

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 4 & 3 & 1 & 6 \end{pmatrix} = \begin{pmatrix} 3 & 5 & 1 & 6 & 4 & 2 \\ 4 & 1 & 2 & 6 & 3 & 5 \end{pmatrix}$$

One-line notation only lists the second row for the natural order of the first row:

$$\pi = \begin{pmatrix} 2 & 5 & 4 & 3 & 1 & 6 \end{pmatrix}$$

Composition

- Permutations of the same set can be composed with function composition.
- Instead of $\sigma \circ \pi$, we write $\sigma\pi$.
- We call $\sigma\pi$ the product of π and σ.
- The product of permutations is a permutation. Why?
- Example:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 4 & 1 & 5 \end{pmatrix} \qquad \pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 5 & 2 & 4 \end{pmatrix}$$

$\sigma\pi =$

$\pi\sigma =$

Cycle Notation – Idea

One-line notation still needs one entry per element and the effect of repeated application is hard to see.

Consider again π with
$\pi(1) = 2, \pi(2) = 5, \pi(3) = 4, \pi(4) = 3, \pi(5) = 1, \pi(6) = 6.$

$$\begin{tikzcd}
1 \arrow[r] & 2 \arrow[rrr, bend right] & 3 \arrow[r] & 4 \arrow[l, bend right] & 5 \arrow[llll, bend right] & 6 \arrow[loop right]
\end{tikzcd}$$

There is a cycle $(1 \quad 2 \quad 5) = (2 \quad 5 \quad 1) = (5 \quad 1 \quad 2)$ and a cycle $(3 \quad 4) = (4 \quad 3)$.

Idea: Write π as product of such cycles.

Cycles

Definition (Cycle)

A permutation σ of finite set S has a *k*-cycle $(e_1 \quad e_2 \quad \dots \quad e_k)$ if

- $e_i \in S$ for $i \in \{1, \dots, k\}$
- $e_i \neq e_j$ for $i \neq j$
- $\sigma(e_i) = e_{i+1}$ for $i \in \{1, \dots, k-1\}$
- $\sigma(e_k) = e_1$

- Don't confuse cycles with permutations in one-line notation.
- A 2-cycle is called a transposition
- A 1-cycle is called a fixed-point of σ.

Cyclic Permutation

Definition (Cyclic Permutation)

A permutation is cyclic if it has a single k-cycle with $k > 1$.

In cycle notation, we represent a cyclic permutation by this cycle.

For example:

Permutation σ of $\{1, \dots, 5\}$ with $\sigma = (1 \quad 3 \quad 4)$ in cycle representation corresponds to

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 4 & 1 & 5 \end{pmatrix}$$

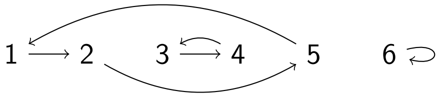
in two-line notation.

Question: Is this representation unique (canonical)?

Cycle Notation – Example

We can write every permutation as a product of disjoint cycles.

Consider again π with
$\pi(1) = 2, \pi(2) = 5, \pi(3) = 4, \pi(4) = 3, \pi(5) = 1, \pi(6) = 6.$



There is a cycle $(1 \quad 2 \quad 5) = (2 \quad 5 \quad 1) = (5 \quad 1 \quad 2)$
and a cycle $(3 \quad 4) = (4 \quad 3)$.

In cycle representation:
$\pi = (1 \quad 2 \quad 5)(3 \quad 4)(6) = (1 \quad 2 \quad 5)(3 \quad 4)$

Cycle Notation – Algorithm

Let π be a permutation of finite set S.

```
function ComputeCycleRepresentation(π, S)
    remaining = S
    cycles = ∅
    while remaining is not empty do
        Remove any element e from remaining.
        Start a new cycle c with e.
        while π(e) ∈ remaining do
            remaining = remaining \ {π(e)}
            Extend c with π(e).
            e = π(e)
        cycles = cycles ∪ {c}
    return cycles
```

The elements of *cycles* can be arranged in any order. $\rightsquigarrow$ Why?

Disjoint Cycles Commute

Theorem

Let $\pi = (e_1 \quad \dots \quad e_n)$ and $\pi' = (e'_1 \quad \dots \quad e'_m)$ be permutations of set S in cycle notation and let π and π' be disjoint, i. e. $e_i \neq e'_j$ for $i \in \{1, \dots, n\}, j \in \{1, \dots, m\}$.

Then $\pi\pi' = \pi'\pi$.

Proof.

Consider an arbitrary element $e \in S$. We distinguish three cases:

If $e = e_i$ for some $i \in \{1, \dots, n\}$ then $\pi(e) = e_j$ for some $j \in \{1, \dots, n\}$. Since the cycles are disjoint, $\pi'(e) = e$ and $\pi'(\pi(e)) = \pi(e)$. Together, this gives $\pi'(\pi(e)) = \pi(\pi'(e))$.

If $e = e'_i$ for some $i \in \{1, \dots, m\}$, we can use the analogous argument to show that $\pi(\pi'(e)) = \pi'(\pi(e))$.

If e occurs in neither cycle then $\pi(e) = e$ and $\pi'(e) = e$, so $\pi'(\pi(e)) = e = \pi(\pi'(e))$. $\square$

In General Cycles Do not Commute

Consider cycles $(1 \quad 2)$ and $(2 \quad 3)$ and set $S = \{1, 2, 3\}$.

$(1 \quad 2)(2 \quad 3) =$

$(2 \quad 3)(1 \quad 2) =$

Transpositions

Theorem

Every cycle can be expressed as a product of transpositions.

Proof idea.

Consider k-cycle $\sigma = (e_1 \quad \dots \quad e_k)$.

We can express σ as $(e_1 \quad e_k)(e_1 \quad e_{k-1}) \dots (e_1 \quad e_2)$. $\square$

Inverse

- Every permutation has an inverse, which is again a permuation.
  - If π is represented in two-line notation, we get π^{-1} by swapping the rows, e. g.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 4 & 1 & 5 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 2 & 4 & 1 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

  - If π is a cycle, we get π^{-1} by reversing the order of the elements, e. g. $(1 \quad 3 \quad 4 \quad 2)^{-1} = (2 \quad 4 \quad 3 \quad 1)$
  - $(\pi\sigma)^{-1} = \sigma^{-1}\pi^{-1}$

Example

$$\sigma = (4\quad 5)(2\quad 3) \qquad \pi = (4\quad 5)(2\quad 1)$$

$$\sigma\pi^{-1} =$$

Another Example

Determine the arrangement of some objects after applying a permutation that operates on the locations.

 and π permutation of $\{1, 2, 3\}$.

Define f with $f($$) = 1$, $f($$) = 2$, $f($$) = 3$
to describe the initial configuration.

Then $\pi \circ f$ describes the resulting configuration.

Last Example

Determine the permutation of locations that leads from one configuration to the other.

Define f with $f($$) = 1$, $f($$) = 2$, $f($$) = 3$
to describe the initial configuration and

function g with $g($$) = 2$, $g($$) = 1$, $g($$) = 3$
for the final configuration.

Then $g \circ f^{-1}$ describes the permutation.