Discrete Mathematics in Computer Science

Partial and Total Functions

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Important Building Blocks of Discrete Mathematics

Important building blocks:

- sets
- relations
- functions

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- functions

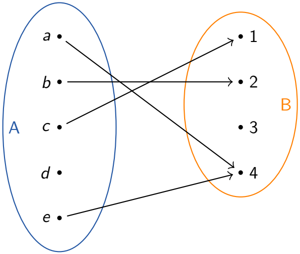
In principle, functions are just a special kind of relations:

- $f : \mathbb{N}_0 \to \mathbb{N}_0$ with $f(x) = x^2$
- relation R over $\mathbb{N}_0$ with $R = \{(x, y) \mid x, y \in \mathbb{N}_0 \text{ and } y = x^2\}$.

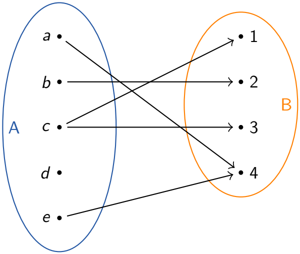
Functional Relations

Definition

A binary relation R over sets A and B is functional
if for every $a \in A$ there is at most one $b \in B$ with $(a, b) \in R$.



functional



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Functions – Examples

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Functions – Examples

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- $abs : \mathbb{Z} \to \mathbb{N}_0$ with
$$abs(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{otherwise} \end{cases}$$
- $distance : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ with
$distance((x_1, y_1), (x_2, y_2)) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Partial Function – Example

Partial function $r : \mathbb{Z} \times \mathbb{Z} \nrightarrow \mathbb{Q}$ with

$$r(n, d) = \begin{cases} \frac{n}{d} & \text{if } d \neq 0 \\ \text{undefined} & \text{otherwise} \end{cases}$$

Partial Functions

Definition (Partial function)

A partial function f from set A to set B (written $f : A \rightharpoonup B$) is given by a functional relation G over A and B.

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We write $f(x) = y$ for $(x, y) \in G$ and say
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If there is no $y \in B$ with $(x, y) \in G$, then $f(x)$ is undefined.

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has graph $\{((n, d), \frac{n}{d}) \mid n \in \mathbb{Z}, d \in \mathbb{Z} \setminus \{0\}\} \subseteq \mathbb{Z}^2 \times \mathbb{Q}$.

Domain (of Definition), Codomain, Image

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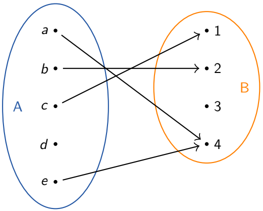

$f : \{a, b, c, d, e\} \nrightarrow \{1, 2, 3, 4\}$
$f(a) = 4, f(b) = 2, f(c) = 1, f(e) = 4$
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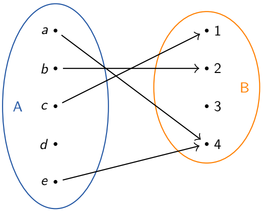

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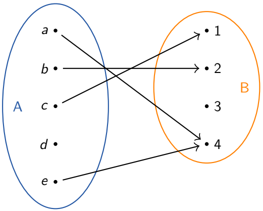
The image (or range) of f is the set
$\mathrm{img}(f) = \{y \mid \text{there is an } x \in A \text{ with } f(x) = y\}$.



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Preimage

The preimage contains all elements of the domain that are mapped to given elements of the codomain.

Definition (Preimage)

Let $f : A \nrightarrow B$ be a partial function and let $Y \subseteq B$.

The preimage of Y under f is the set $f^{-1}[Y] = \{x \in A \mid f(x) \in Y\}$.

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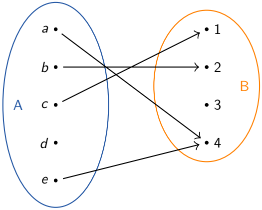

$f^{-1}[\{1\}] =$

$f^{-1}[\{3\}] =$

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$f^{-1}[\{1, 2\}] =$

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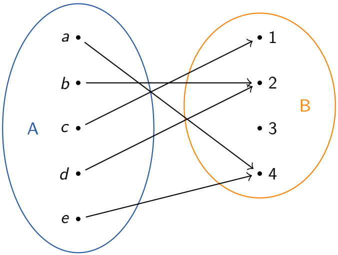

Specifying a Function

Some common ways of specifying a function:

- Listing the mapping explicitly, e. g.
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- In terms of other functions, e. g. inverse, composition

Relationship to Functions in Programming

```python
def factorial(n):
    if n == 0:
        return 1
    else:
        return n * factorial(n-1)
```

→ Relationship between recursion and recurrence

Relationship to Functions in Programming

```
def foo(n):
    value = ...
    while <some condition>:
        ...
        value = ...
    return value
```

→ Does possibly not terminate on all inputs.
→ Value is undefined for such inputs.
→ Theoretical computer science: partial function

Relationship to Functions in Programming

```python
import random
counter = 0

def bar(n):
    print("Hi! I got input", n)
    global counter
    counter += 1
    return random.choice([1,2,n])
```

→ Functions in programming don't always compute mathematical functions (except *purely functional languages*).

→ In addition, not all mathematical functions are computable.

Discrete Mathematics in Computer Science

Operations on Partial Functions

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Restrictions and Extensions

Definition (restriction and extension)

Let $f : A \nrightarrow B$ be a partial function and let $X \subseteq A$.
The restriction of f to X is the partial function $f|_X : X \nrightarrow B$ with $f|_X(x) = f(x)$ for all $x \in X$.

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What's the graph of the restriction?

What's the restriction of f to its domain?

Function Composition

Definition (Composition of partial functions)

Let $f : A \nrightarrow B$ and $g : B \nrightarrow C$ be partial functions.

The composition of f and g is $g \circ f : A \nrightarrow C$ with

$$(g \circ f)(x) = \begin{cases} g(f(x)) & \text{if } f \text{ is defined for } x \text{ and} \\ & g \text{ is defined for } f(x) \\ \text{undefined} & \text{otherwise} \end{cases}$$

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If f and g are functions, their composition is a function.

Example:

$$\begin{aligned} f : \mathbb{N}_0 \to \mathbb{N}_0 &\quad \text{with } f(x) = x^2 \\ g : \mathbb{N}_0 \to \mathbb{N}_0 &\quad \text{with } g(x) = x + 3 \\ (g \circ f)(x) = & \end{aligned}$$

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  - $(g \circ f)(x) = x^2 + 3$
  - $(f \circ g)(x) = (x + 3)^2$
- associative, i. e. $h \circ (g \circ f) = (h \circ g) \circ f$
  $\to$ analogous to associativity of relation composition

Function Composition in Programming

We implicitly compose functions all the time...

```
def foo(n):
    ...
    x = somefunction(n)
    y = someotherfunction(x)
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Many languages also allow explicit composition of functions, e. g. in Haskell:

```haskell
incr x = x + 1
square x = x * x
squareplusone = incr . square
```

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Properties of Functions

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  total functions map it to exactly one such value.
- Different elements of the domain can have the same image.
- There can be values of the codomain
  that aren't the image of any element of the domain.
- We often want to exclude such cases
  $\rightarrow$ define additional properties to say this quickly

Injective Functions

An injective function maps distinct elements of its domain to distinct elements of its co-domain.

Definition (Injective Function)

A function $f : A \to B$ is injective (also one-to-one or an injection) if for all $x, y \in A$ with $x \neq y$ it holds that $f(x) \neq f(y)$.

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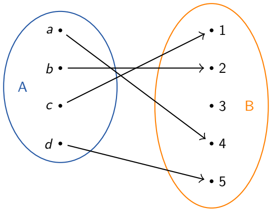
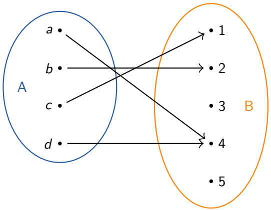


injective



not injective

Injective Functions – Examples

Which of these functions are injective?

- $f : \mathbb{Z} \to \mathbb{N}_0$ with $f(x) = |x|$
- $g : \mathbb{N}_0 \to \mathbb{N}_0$ with $g(x) = x^2$
- $h : \mathbb{N}_0 \to \mathbb{N}_0$ with $h(x) = \begin{cases} x - 1 & \text{if } x \text{ is odd} \\ x + 1 & \text{if } x \text{ is even} \end{cases}$

Composition of Injective Functions

Theorem

If $f : A \to B$ and $g : B \to C$ are injective functions then also $g \circ f$ is injective.

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Proof.

Consider arbitrary elements $x, y \in A$ with $x \neq y$.
Since f is injective, we know that $f(x) \neq f(y)$.
As g is injective, this implies that $g(f(x)) \neq g(f(y))$.
With the definition of $g \circ f$, we conclude that
$(g \circ f)(x) \neq (g \circ f)(y)$.
Overall, this shows that $g \circ f$ is injective. $\square$

Surjective Functions

A surjective function maps at least one elements to every element of its co-domain.

Definition (Surjective Function)

A function $f : A \to B$ is surjective (also onto or a surjection) if its image is equal to its codomain,
i. e. for all $y \in B$ there is an $x \in A$ with $f(x) = y$.

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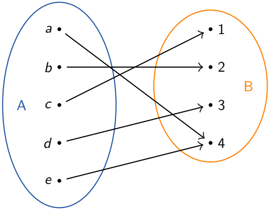

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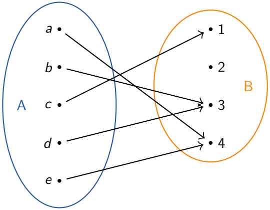

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Proof.

Consider an arbitary element $z \in C$.
Since g is surjective, there is a $y \in B$ with $g(y) = z$.
As f is surjective, for such a y there is an $x \in A$ with $f(x) = y$ and thus $g(f(x)) = z$.
Overall, for every $z \in C$ there is an $x \in A$ with $(g \circ f)(x) = g(f(x)) = z$, so $g \circ f$ is surjective. $\square$

Bijective Functions

A bijective function pairs every element of its domain with exactly one element of its codomain and every element of the codomain is paired with exactly one element of the domain.

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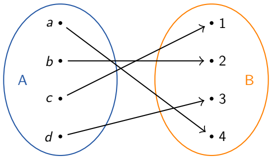

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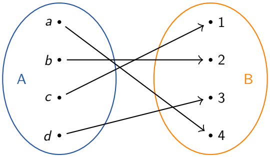
Definition (Bijective Function)

A function is bijective (also a one-to-one correspondence or a bijection) if it is injective and surjective.



bijection

Corollary

The composition of two bijective functions is bijective.

Bijective Functions – Examples

Which of these functions are bijective?

- $f : \mathbb{Z} \to \mathbb{N}_0$ with $f(x) = |x|$
- $g : \mathbb{N}_0 \to \mathbb{N}_0$ with $g(x) = x^2$
- $h : \mathbb{N}_0 \to \mathbb{N}_0$ with $h(x) = \begin{cases} x - 1 & \text{if } x \text{ is odd} \\ x + 1 & \text{if } x \text{ is even} \end{cases}$

Inverse Function

Definition

Let $f : A \to B$ be a bijection.
The inverse function of f is the function $f^{-1} : B \to A$ with $f^{-1}(y) = x$ iff $f(x) = y$.

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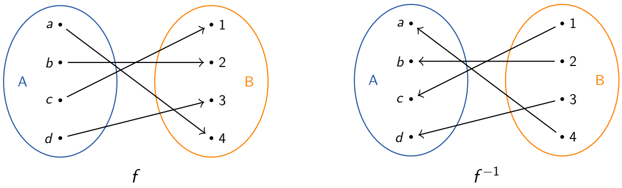

Inverse Function and Composition

Theorem

Let $f : A \to B$ be a bijection.

1. *For all $x \in A$ it holds that $f^{-1}(f(x)) = x$.*
2. *For all $y \in B$ it holds that $f(f^{-1}(y)) = y$.*
3. $(f^{-1})^{-1} = f$

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Proof sketch.

1. For $x \in A$ let $y = f(x)$. Then $f^{-1}(f(x)) = f^{-1}(y) = x$

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Proof sketch.

1. For $x \in A$ let $y = f(x)$. Then $f^{-1}(f(x)) = f^{-1}(y) = x$
2. For $y \in B$ there is exactly one x with $y = f(x)$. With this x it holds that $f^{-1}(y) = x$ and overall $f(f^{-1}(y)) = f(x) = y$.

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Proof sketch.

1. For $x \in A$ let $y = f(x)$. Then $f^{-1}(f(x)) = f^{-1}(y) = x$
2. For $y \in B$ there is exactly one x with $y = f(x)$. With this x it holds that $f^{-1}(y) = x$ and overall $f(f^{-1}(y)) = f(x) = y$.
3. Def. of inverse: $(f^{-1})^{-1}(x) = y$ iff $f^{-1}(y) = x$ iff $f(x) = y$.

Inverse Function

Theorem

Let $f : A \to B$ *and* $g : B \to C$ *be bijections.*
Then $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Inverse Function

Theorem

Let $f : A \to B$ and $g : B \to C$ be bijections.
Then $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Proof.

We need to show that for all $x \in C$ it holds that
$(g \circ f)^{-1}(x) = (f^{-1} \circ g^{-1})(x)$.

Consider an arbitrary $x \in C$ and let $y = (g \circ f)^{-1}(x)$.
By the definition of the inverse $(g \circ f)(y) = x$.

Let $z = f(y)$. With $(g \circ f)(y) = g(f(y))$, we know that $x = g(z)$.
From $z = f(y)$ we get $f^{-1}(z) = y$ and
from $x = g(z)$ we get $g^{-1}(x) = z$.

This gives $(f^{-1} \circ g^{-1})(x) = f^{-1}(g^{-1}(x)) = f^{-1}(z) = y$. $\square$

Summary

- injective function: maps distinct elements of its domain to distinct elements of its co-domain.
- surjective function: maps at least one element to every element of its co-domain.
- bijective function: injective and surjective
  $\rightarrow$ one-to-one correspondence
- Bijective functions are invertible. The inverse function of f maps the image of x under f to x.