Discrete Mathematics in Computer Science

B8. Functions

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B8.1 Partial and Total Functions

Important Building Blocks of Discrete Mathematics

Important building blocks:

- sets
- relations
- functions

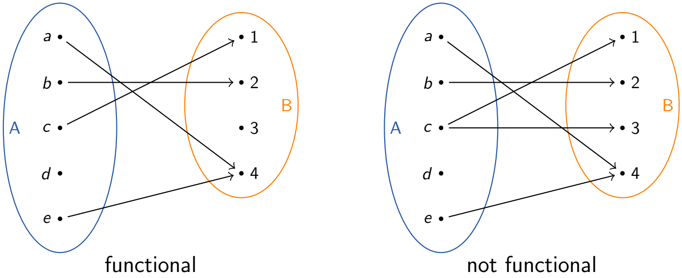
In principle, functions are just a special kind of relations:

- $f : \mathbb{N}_0 \to \mathbb{N}_0$ with $f(x) = x^2$
- relation R over $\mathbb{N}_0$ with $R = \{(x, y) \mid x, y \in \mathbb{N}_0 \text{ and } y = x^2\}$.

Functional Relations

Definition

A binary relation R over sets A and B is functional if for every $a \in A$ there is at most one $b \in B$ with $(a, b) \in R$.



Functions – Examples

- $f : \mathbb{N}_0 \to \mathbb{N}_0$ with $f(x) = x^2 + 1$
- $abs : \mathbb{Z} \to \mathbb{N}_0$ with
$$abs(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{otherwise} \end{cases}$$
- $distance : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ with
$distance((x_1, y_1), (x_2, y_2)) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Partial Function – Example

Partial function $r : \mathbb{Z} \times \mathbb{Z} \nrightarrow \mathbb{Q}$ with

$$r(n, d) = \begin{cases} \frac{n}{d} & \text{if } d \neq 0 \\ \text{undefined} & \text{otherwise} \end{cases}$$

Partial Functions

Definition (Partial function)

A partial function f from set A to set B (written $f : A \nrightarrow B$) is given by a functional relation G over A and B.

Relation G is called the graph of f.

We write $f(x) = y$ for $(x, y) \in G$ and say y is the image of x under f.

If there is no $y \in B$ with $(x, y) \in G$, then $f(x)$ is undefined.

Partial function $r : \mathbb{Z} \times \mathbb{Z} \nrightarrow \mathbb{Q}$ with

$$r(n, d) = \begin{cases} \frac{n}{d} & \text{if } d \neq 0 \\ \text{undefined} & \text{otherwise} \end{cases}$$

has graph $\{((n, d), \frac{n}{d}) \mid n \in \mathbb{Z}, d \in \mathbb{Z} \setminus \{0\}\} \subseteq \mathbb{Z}^2 \times \mathbb{Q}$.

Domain (of Definition), Codomain, Image

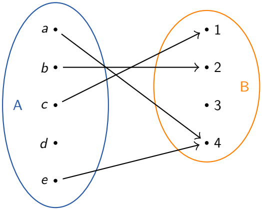
Definition (domain of definition, codomain, image)

Let $f : A \nrightarrow B$ be a partial function.

Set A is called the domain of f, set B is its codomain.

The domain of definition of f is the set
$\text{dom}(f) = \{x \in A \mid \text{there is a } y \in B \text{ with } f(x) = y\}$.

The image (or range) of f is the set
$\text{img}(f) = \{y \mid \text{there is an } x \in A \text{ with } f(x) = y\}$.



$f : \{a, b, c, d, e\} \nrightarrow \{1, 2, 3, 4\}$
$f(a) = 4, f(b) = 2, f(c) = 1, f(e) = 4$
domain $\{a, b, c, d, e\}$
codomain $\{1, 2, 3, 4\}$
domain of definition $\text{dom}(f) = \{a, b, c, e\}$
image $\text{img}(f) = \{1, 2, 4\}$

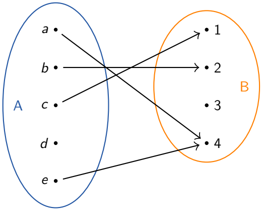
Preimage

The preimage contains all elements of the domain that are mapped to given elements of the codomain.

Definition (Preimage)

Let $f : A \nrightarrow B$ be a partial function and let $Y \subseteq B$.

The preimage of Y under f is the set $f^{-1}[Y] = \{x \in A \mid f(x) \in Y\}$.



$f^{-1}[\{1\}] =$

$f^{-1}[\{3\}] =$

$f^{-1}[\{4\}] =$

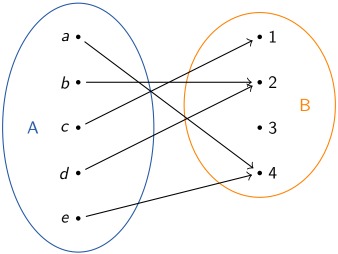
$f^{-1}[\{1, 2\}] =$

Total Functions

Definition (Total function)

A (total) function $f : A \to B$ from set A to set B is a partial function from A to B such that $f(x)$ is defined for all $x \in A$.

$\rightarrow$ no difference between the domain and the domain of definition



Specifying a Function

Some common ways of specifying a function:

- Listing the mapping explicitly, e. g.
  $f(a) = 4, f(b) = 2, f(c) = 1, f(e) = 4$ or
  $f = \{a \mapsto 4, b \mapsto 2, c \mapsto 1, e \mapsto 4\}$
- By a formula, e. g. $f(x) = x^2 + 1$
- By recurrence, e. g.
  $0! = 1$ and
  $n! = n(n-1)!$ for $n > 0$
- In terms of other functions, e. g. inverse, composition

Relationship to Functions in Programming

```
def factorial(n):
    if n == 0:
        return 1
    else:
        return n * factorial(n-1)
```

→ Relationship between recursion and recurrence

Relationship to Functions in Programming

```
def foo(n):
    value = ...
    while <some condition>:
        ...
        value = ...
    return value
```

→ Does possibly not terminate on all inputs.
→ Value is undefined for such inputs.
→ Theoretical computer science: partial function

Relationship to Functions in Programming

```
import random
counter = 0

def bar(n):
    print("Hi! I got input", n)
    global counter
    counter += 1
    return random.choice([1,2,n])
```

→ Functions in programming don't always compute mathematical functions (except *purely functional languages*).

→ In addition, not all mathematical functions are computable.

B8.2 Operations on Partial Functions

Restrictions and Extensions

Definition (restriction and extension)

Let $f : A \nrightarrow B$ be a partial function and let $X \subseteq A$.
The restriction of f to X is the partial function $f|_X : X \nrightarrow B$ with $f|_X(x) = f(x)$ for all $x \in X$.

A function $f' : A' \nrightarrow B$ is called an extension of f if $A \subseteq A'$ and $f'|_A = f$.

The restriction of f to its domain of definition is a total function.

What's the graph of the restriction?

What's the restriction of f to its domain?

Function Composition

Definition (Composition of partial functions)

Let $f : A \nrightarrow B$ and $g : B \nrightarrow C$ be partial functions.

The composition of f and g is $g \circ f : A \nrightarrow C$ with

$$(g \circ f)(x) = \begin{cases} g(f(x)) & \text{if } f \text{ is defined for } x \text{ and} \\ & g \text{ is defined for } f(x) \\ \text{undefined} & \text{otherwise} \end{cases}$$

Corresponds to relation composition of the graphs.

If f and g are functions, their composition is a function.

Example:

$$f : \mathbb{N}_0 \rightarrow \mathbb{N}_0 \quad \text{with } f(x) = x^2$$
$$g : \mathbb{N}_0 \rightarrow \mathbb{N}_0 \quad \text{with } g(x) = x + 3$$
$$(g \circ f)(x) =$$

Properties of Function Composition

Function composition is

- not commutative:
  - $f : \mathbb{N}_0 \to \mathbb{N}_0$ with $f(x) = x^2$
  - $g : \mathbb{N}_0 \to \mathbb{N}_0$ with $g(x) = x + 3$
  - $(g \circ f)(x) = x^2 + 3$
  - $(f \circ g)(x) = (x + 3)^2$
- associative, i. e. $h \circ (g \circ f) = (h \circ g) \circ f$
  $\rightarrow$ analogous to associativity of relation composition

Function Composition in Programming

We implicitly compose functions all the time...

```
def foo(n):
    ...
    x = somefunction(n)
    y = someotherfunction(x)
    ...
```

Many languages also allow explicit composition of functions, e. g. in Haskell:

```
incr x = x + 1
square x = x * x
squareplusone = incr . square
```

B8.3 Properties of Functions

Properties of Functions

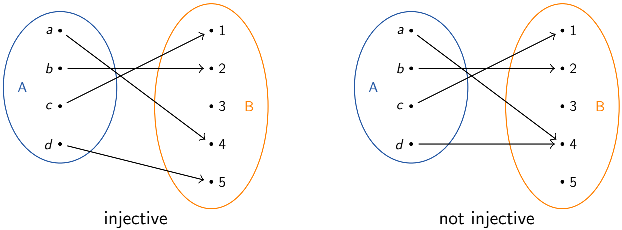
- Partial functions map every element of their domain to at most one element of their codomain, total functions map it to exactly one such value.
- Different elements of the domain can have the same image.
- There can be values of the codomain that aren't the image of any element of the domain.
- We often want to exclude such cases
  $\rightarrow$ define additional properties to say this quickly

Injective Functions

An injective function maps distinct elements of its domain to distinct elements of its co-domain.

Definition (Injective Function)

A function $f : A \to B$ is injective (also one-to-one or an injection) if for all $x, y \in A$ with $x \neq y$ it holds that $f(x) \neq f(y)$.



Injective Functions – Examples

Which of these functions are injective?

- $f : \mathbb{Z} \to \mathbb{N}_0$ with $f(x) = |x|$
- $g : \mathbb{N}_0 \to \mathbb{N}_0$ with $g(x) = x^2$
- $h : \mathbb{N}_0 \to \mathbb{N}_0$ with $h(x) = \begin{cases} x - 1 & \text{if } x \text{ is odd} \\ x + 1 & \text{if } x \text{ is even} \end{cases}$

Composition of Injective Functions

Theorem

If $f : A \to B$ and $g : B \to C$ are injective functions then also $g \circ f$ is injective.

Proof.

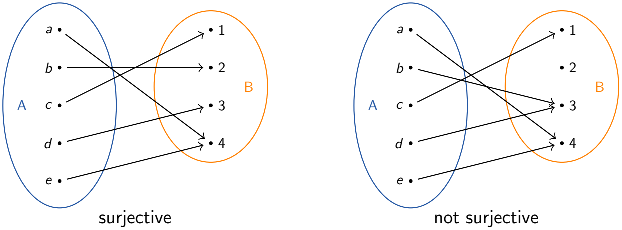
Consider arbitrary elements $x, y \in A$ with $x \neq y$.
Since f is injective, we know that $f(x) \neq f(y)$.
As g is injective, this implies that $g(f(x)) \neq g(f(y))$.
With the definition of $g \circ f$, we conclude that $(g \circ f)(x) \neq (g \circ f)(y)$.
Overall, this shows that $g \circ f$ is injective. □

Surjective Functions

A surjective function maps at least one elements to every element of its co-domain.

Definition (Surjective Function)

A function $f : A \to B$ is surjective (also onto or a surjection) if its image is equal to its codomain, i. e. for all $y \in B$ there is an $x \in A$ with $f(x) = y$.



Surjective Functions – Examples

Which of these functions are surjective?

- $f : \mathbb{Z} \to \mathbb{N}_0$ with $f(x) = |x|$
- $g : \mathbb{N}_0 \to \mathbb{N}_0$ with $g(x) = x^2$
- $h : \mathbb{N}_0 \to \mathbb{N}_0$ with $h(x) = \begin{cases} x - 1 & \text{if } x \text{ is odd} \\ x + 1 & \text{if } x \text{ is even} \end{cases}$

Composition of Surjective Functions

Theorem

If $f : A \to B$ and $g : B \to C$ are surjective functions then also $g \circ f$ is surjective.

Proof.

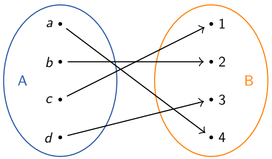
Consider an arbitary element $z \in C$.
Since g is surjective, there is a $y \in B$ with $g(y) = z$.
As f is surjective, for such a y there is an $x \in A$ with $f(x) = y$ and thus $g(f(x)) = z$.
Overall, for every $z \in C$ there is an $x \in A$ with $(g \circ f)(x) = g(f(x)) = z$, so $g \circ f$ is surjective. □

Bijective Functions

A bijective function pairs every element of its domain with exactly one element of its codomain and every element of the codomain is paired with exactly one element of the domain.

Definition (Bijective Function)

A function is bijective (also a one-to-one correspondence or a bijection) if it is injective and surjective.



bijection

Corollary

The composition of two bijective functions is bijective.

Bijective Functions – Examples

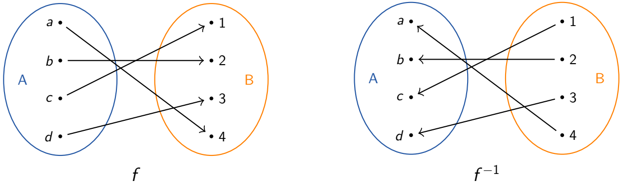
Which of these functions are bijective?

- $f : \mathbb{Z} \to \mathbb{N}_0$ with $f(x) = |x|$
- $g : \mathbb{N}_0 \to \mathbb{N}_0$ with $g(x) = x^2$
- $h : \mathbb{N}_0 \to \mathbb{N}_0$ with $h(x) = \begin{cases} x - 1 & \text{if } x \text{ is odd} \\ x + 1 & \text{if } x \text{ is even} \end{cases}$

Inverse Function

Definition

Let $f : A \to B$ be a bijection.
The inverse function of f is the function $f^{-1} : B \to A$ with $f^{-1}(y) = x$ iff $f(x) = y$.



Inverse Function and Composition

Theorem

Let $f : A \to B$ be a bijection.

1. *For all $x \in A$ it holds that $f^{-1}(f(x)) = x$.*
2. *For all $y \in B$ it holds that $f(f^{-1}(y)) = y$.*
3. $(f^{-1})^{-1} = f$

Proof sketch.

1. For $x \in A$ let $y = f(x)$. Then $f^{-1}(f(x)) = f^{-1}(y) = x$
2. For $y \in B$ there is exactly one x with $y = f(x)$. With this x it holds that $f^{-1}(y) = x$ and overall $f(f^{-1}(y)) = f(x) = y$.
3. Def. of inverse: $(f^{-1})^{-1}(x) = y$ iff $f^{-1}(y) = x$ iff $f(x) = y$.

Inverse Function

Theorem

Let $f : A \to B$ and $g : B \to C$ be bijections.

Then $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Proof.

We need to show that for all $x \in C$ it holds that $(g \circ f)^{-1}(x) = (f^{-1} \circ g^{-1})(x)$.

Consider an arbitrary $x \in C$ and let $y = (g \circ f)^{-1}(x)$.
By the definition of the inverse $(g \circ f)(y) = x$.

Let $z = f(y)$. With $(g \circ f)(y) = g(f(y))$, we know that $x = g(z)$.
From $z = f(y)$ we get $f^{-1}(z) = y$ and
from $x = g(z)$ we get $g^{-1}(x) = z$.

This gives $(f^{-1} \circ g^{-1})(x) = f^{-1}(g^{-1}(x)) = f^{-1}(z) = y$. □

Summary

- injective function: maps distinct elements of its domain to distinct elements of its co-domain.
- surjective function: maps at least one element to every element of its co-domain.
- bijective function: injective and surjective
  $\rightarrow$ one-to-one correspondence
- Bijective functions are invertible. The inverse function of f maps the image of x under f to x.