Discrete Mathematics in Computer Science

B6. Equivalence and Order Relations

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B6.1 Equivalence Relations and Partitions

Relations: Recap

- A relation over sets $S_1, \dots, S_n$ is a set $R \subseteq S_1 \times \dots \times S_n$.
- Possible properties of homogeneous relations R over S:
  - reflexive: $(x, x) \in R$ for all $x \in S$
  - irreflexive: $(x, x) \notin R$ for all $x \in S$
  - symmetric: $(x, y) \in R$ iff $(y, x) \in R$
  - asymmetric: if $(x, y) \in R$ then $(y, x) \notin R$
  - antisymmetric: if $(x, y) \in R$ then $(y, x) \notin R$ or $x = y$
  - transitive: if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$

Motivation

- Think of any attribute that two objects can have in common, e. g. their color.
- We could place the objects into distinct "buckets", e. g. one bucket for each color.
- We also can define a relation $\sim$ such that $x \sim y$ iff x and y share the attribute, e. g.have the same color.
- Would this relation be
  - reflexive?
  - irreflexive?
  - symmetric?
  - asymmetric?
  - antisymmetric?
  - transitive?

Equivalence Relation

Definition (Equivalence Relation)

A binary relation $\sim$ over set S is an equivalence relation if $\sim$ is reflexive, symmetric and transitive.

Is this definition indeed what we want?
Does it allow us to partition the objects into buckets (e. g. one group for all objects that share a specific color)?

Partition

Definition (Partition)

A partition of a set S is a set $P \subseteq \mathcal{P}(S)$ such that

- $X \neq \emptyset$ for all $X \in P$,
- $\bigcup_{X \in P} X = S$, and
- $X \cap Y = \emptyset$ for all $X, Y \in P$ with $X \neq Y$,

The elements of P are called the blocks of the partition.

Partition

Let $S = \{e_1, \dots, e_5\}$.

Which of the following sets are partitions of S?

- $P_1 = \{\{e_1, e_4\}, \{e_3\}, \{e_2, e_5\}\}$
- $P_2 = \{\{e_1, e_4, e_5\}, \{e_3\}\}$
- $P_3 = \{\{e_1, e_4, e_5\}, \{e_3\}, \{e_2, e_5\}\}$
- $P_4 = \{\{e_1\}, \{e_2\}, \{e_3\}, \{e_4\}, \{e_5\}\}$
- $P_5 = \{\{e_1\}, \{e_2\}, \{e_3\}, \{e_4\}, \{e_5\}, \{\}\}$

A Property of Partitions

Lemma

Let S be a set and P be a partition of S.
Then every $x \in S$ is an element of exactly one $X \in P$.

Proof: $\rightsquigarrow$ exercises

Block of an Element

The lemma enables the following definition:

Definition

Let S be a set and P be a partition of S.

For $e \in S$ we denote by $[e]_P$ the block $X \in P$ such that $e \in X$.

Consider partition $P = \{\{e_1, e_4\}, \{e_3\}, \{e_2, e_5\}\}$ of $\{e_1, \dots, e_5\}$.

$[e_1]_P =$

Connection between Partitions and Equivalence Relations?

- We will now explore the connection between partitions and equivalence relations.
- Spoiler: They are essentially the same concept.

Partitions Induce Equivalence Relations I

Definition (Relation induced by a partition)

Let S be a set and P be a partition of S.

The relation $\sim_P$ induced by P is the binary relation over S with

$$x \sim_P y \text{ iff } [x]_P = [y]_P.$$

$x \sim_P y$ iff x and y are in the same block of P.

Consider partition $P = \{\{1,4,5\},\{2,3\}\}$ of set $\{1,2,\dots,5\}$.

$$\begin{aligned}\sim_P = \{&(1,1),(1,4),(1,5),(4,1),(4,4),(4,5),(5,1),(5,4),(5,5),\\ &(2,2),(2,3),(3,2),(3,3)\}\end{aligned}$$

We will show that $\sim_P$ is an equivalence relation.

Partitions Induce Equivalence Relations II

Theorem

Let P be a partition of S.
Relation $\sim_P$ induced by P is an equivalence relation over S.

Proof.

We need to show that $\sim_P$ is reflexive, symmetric and transitive.

reflexive: As $=$ is reflexive it holds for all $x \in S$ that $[x]_P = [x]_P$ and hence also that $x \sim_P x$.

symmetric: If $x \sim_P y$ then $[x]_P = [y]_P$. With the symmetry of $=$ we get that $[y]_P = [x]_P$ and conclude that $y \sim_P x$.

transitive: If $x \sim_P y$ and $y \sim_P z$ then $[x]_P = [y]_P$ and $[y]_P = [z]_P$. As $=$ is transitive, it then also holds that $[x]_P = [z]_P$ and hence $x \sim_P z$. □

Equivalence Classes

Definition (equivalence class)

Let R be an equivalence relation over set S.

For any $x \in S$, the equivalence class of x is the set

$$[x]_R = \{y \in S \mid xRy\}.$$

Consider
$R = \{(1,1),(1,4),(1,5),(4,1),(4,4),(4,5),(5,1),(5,4),(5,5),$
$(2,2),(2,3),(3,2),(3,3)\}$
over set $\{1,2,\dots,5\}$.

$[4]_R =$

Equivalence Relations Induce Partitions

Theorem

Let R be an equivalence relation over set S.
The set $P = \{[x]_R \mid x \in S\}$ is a partition of S.

Proof.

We need to show that

1. $X \neq \emptyset$ for all $X \in P$,
2. $\bigcup_{X \in P} X = S$, and
3. $X \cap Y = \emptyset$ for all $X, Y \in P$ with $X \neq Y$,

1) For $x \in S$, it holds that $x \in [x]_R$ because R is reflexive.
Hence, no $X \in P$ is empty. . . .

Equivalence Relations Induce Partitions

Proof (continued).

For 2) we show $\bigcup_{X \in P} X \subseteq S$ and $\bigcup_{X \in P} X \supseteq S$ separately.

$\subseteq$: Consider an arbitrary $x \in \bigcup_{X \in P} X$. Since x is contained in the union, it must be an element of some $X \in P$. Consider such an X. By the definition of P, there is a $y \in S$ such that $X = [y]_R$.
Since $x \in [y]_R$, it holds that yRx.
As R is a relation over S, this implies that $x \in S$.

$\supseteq$: Consider an arbitrary $x \in S$. Since $x \in [x]_R$ (cf. 1) and $[x]_R \in P$, it holds that $x \in \bigcup_{X \in P} X$. ...

Equivalence Relations Induce Partitions

Proof (continued).

We show 3) by contrapositive:
For all $X, Y \in P$: if $X \cap Y \neq \emptyset$ then $X = Y$.

Let X, Y be two sets from P with $X \cap Y \neq \emptyset$.
Then there is an e with $e \in X \cap Y$ and there are $x, y \in S$ with $X = [x]_R$ and $Y = [y]_R$. Consider such e, x, y.

As $e \in [x]_R$ and $e \in [y]_R$ it holds that xRe and yRe. Since R is symmetric, we get from yRe that eRy. By transitivity, xRe and eRy imply xRy, which by symmetry also gives yRx.

We show $[x]_R \subseteq [y]_R$: consider an arbitrary $z \in [x]_R$. Then xRz. From yRx and xRz, by transitivity we get yRz. This establishes $z \in [y]_R$. As z was chosen arbitarily, it holds that $[x]_R \subseteq [y]_R$.

Analogously, we can show that $[x]_R \supseteq [y]_R$, so overall $X = Y$. $\square$

Summary

- We typically encounter equivalence relations when we consider objects as equivalent wrt. some attribute/property.
- A relation is an equivalence relation if it is reflexive, symmetric and transitive.
- A partition of a set groups the elements into non-empty subsets.
- The concepts are closely connected: in principle just different perspectives on the same "situation".

B6.2 Partial Orders

Order Relations

- An equivalence relation is reflexive, symmetric and transitive.
- Such a relation induces a partition into "equivalent" objects.
- We now consider other combinations of properties, that allow us to compare objects in a set against other objects.
- "Number x is not larger than number y."
  "Set S is a subset of set T."
  "Jerry runs at least as fast as Tom."
  "Pasta tastes better than Potatoes."

Partial Orders

- We begin with partial orders.
- Example partial order relations are $\leq$ over $\mathbb{N}$ or $\subseteq$ for sets.
- Are these relations
  - reflexive?
  - irreflexive?
  - symmetric?
  - asymmetric?
  - antisymmetric?
  - transitive?

Partial Orders – Definition

Definition (Partial order, partially ordered sets)

A binary relation $\preceq$ over set S is a partial order if $\preceq$ is reflexive, antisymmetric and transitive.

A partially ordered set (or poset) is a pair (S, R) where S is a set and R is a partial order over S.

Which of these relations are partial orders?

- strict subset relation $\subset$ for sets
- not-less-than relation $\geq$ over $\mathbb{N}_0$
- $R = \{(a, a), (a, b), (b, b), (b, c), (c, c)\}$ over $\{a, b, c\}$

Least and Greatest Element

Some special elements of posets:

Definition (Least and greatest element)

Let $\preceq$ be a partial order over set S.

An element $x \in S$ is the least element of S if for all $y \in S$ it holds that $x \preceq y$.

It is the greatest element of S if for all $y \in S$, $y \preceq x$.

- Is there a least/greatest element? Which one?
  - $S = \{1, 2, 3\}$ and $\preceq\ = \{(x, y) \mid x, y \in S \text{ and } x \leq y\}$.
  - $\mathbb{N}_0$ and standard relation $\leq$.
- Why can we say the least element instead of a least element?

Uniqueness of Least Element

Theorem

Let $\preceq$ be a partial order over set S.
If S contains a least element, it contains exactly one least element.

Proof.

By contradiction: Assume x, y are least elements of S with $x \neq y$.
Since x is a least element, $x \preceq y$ is true.
Since y is a least element, $y \preceq x$ is true.
As a partial order is antisymmetric, this implies that $x = y$. $\lightning$ $\square$

Analogously: If there is a greatest element then is unique.

Minimal and Maximal Elements

Definition (Minimal/Maximal element of a set)

Let $\preceq$ be a partial order over set S.
An element $x \in S$ is a minimal element of S
if there is no $y \in S$ with $y \preceq x$ and $x \neq y$.

An element $x \in S$ is a maximal element of S
if there is no $y \in S$ with $x \preceq y$ and $x \neq y$.

A set can have several minimal elements and no least element.
Example?

Total Orders

- Relations $\leq$ over $\mathbb{N}_0$ and $\subseteq$ for sets are partial orders.
- Can we compare every object against every object?
  - For all $x, y \in \mathbb{N}_0$ it holds that $x \leq y$ or that $y \leq x$ (or both).
  - $\{1, 2\} \not\subseteq \{2, 3\}$ and $\{2, 3\} \not\subseteq \{1, 2\}$
- Relation $\leq$ is a total order, relation $\subseteq$ is not.

Total Order – Definition

Definition (Total relation)

A binary relation R over set S is total (or connex) if for all $x, y \in S$ at least one of xRy or yRx is true.

Definition (Total order)

A binary relation is a total order if it is total and a partial order.

Summary

- A partial order is reflexive, antisymmetric and transitive.
- With a total order $\preceq$ over S there are no elements $x, y \in S$ with $x \not\preceq y$ and $y \not\preceq x$.
- If x is the greatest element of a set S, it is greater than every element: for all $y \in S$ it holds that $y \preceq x$.
- If x is a maximal element of set S then it is not smaller than any other element y: there is no $y \in S$ with $x \preceq y$ and $x \neq y$.

B6.3 Strict Orders

Strict Orders

- A partial order is reflexive, antisymmetric and transitive.
- We now consider strict orders.
- Example strict order relations are $<$ over $\mathbb{N}$ or $\subset$ for sets.
- Are these relations
  - reflexive?
  - irreflexive?
  - symmetric?
  - asymmetric?
  - antisymmetric?
  - transitive?

Strict Orders – Definition

Definition (Strict order)

A binary relation $\prec$ over set S is a strict order if $\prec$ is irreflexive, asymmetric and transitive.

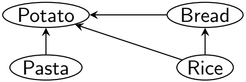
Which of these relations are strict orders?

- subset relation $\subseteq$ for sets
- strict superset relation $\supset$ for sets

Can a relation be both, a partial order and a strict order?

Strict Total Orders

- As partial orders, a strict order does not automatically allow us to rank arbitrary two objects against each other.
- Example 1 (personal preferences):
  - "Pasta tastes better than potato."
  - "Rice tastes better than bread."
  - "Bread tastes better than potato."
  - "Rice tastes better than potato."



  - This definition of "tastes better than" is a strict order.
  - No ranking of pasta against rice or of pasta against bread.
- Example 2: $\subset$ relation for sets
- It doesn't work to simply require that the strict order is total. Why?

Strict Total Orders – Definition

Definition (Trichotomy)

A binary relation R over set S is trichotomous if for all $x, y \in S$ exactly one of xRy, yRx or $x = y$ is true.

Definition (Strict total order)

A binary relation $\prec$ over S is a strict total order if $\prec$ is trichotomous and a strict order.

A strict total order completely ranks the elements of set S.

Example: $<$ relation over $\mathbb{N}_0$ gives the standard ordering $0, 1, 2, 3, \ldots$ of natural numbers.

Special Elements

Special elements are defined almost as for partial orders:

Definition (Least/greatest/minimal/maximal element of a set)

Let $\prec$ be a strict order over set S.

An element $x \in S$ is the least element of S
if for all $y \in S$ where $y \neq x$ it holds that $x \prec y$.

It is the greatest element of S if for all $y \in S$ where $y \neq x$, $y \prec x$.

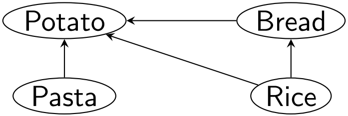
Element $x \in S$ is a minimal element of S
if there is no $y \in S$ with $y \prec x$.

It is a maximal element of S
if there is no $y \in S$ with $x \prec y$.

Special Elements – Example

Consider again the previous example:

$S = \{\text{Pasta}, \text{Potato}, \text{Bread}, \text{Rice}\}$
$\prec\ = \{(\text{Pasta}, \text{Potato}), (\text{Bread}, \text{Potato}),$
$(\text{Rice}, \text{Potato}), (\text{Rice}, \text{Bread})\}$



Is there a least and a greatest element?
Which elements are maximal or minimal?

Summary and Outlook

- A strict order is irreflexive, asymmetric and transitive.
- Strict total orders and special elements are analogously defined as for partial sets but with a special treatment of equal elements.
- For partial order $\preceq$ we can define a related strict order $\prec$ as $x \prec y$ if $x \preceq y$ and $y \not\preceq x$.
- For strict order $\prec$ we can define a related partial order $\preceq$ as $x \preceq y$ if $x \prec y$ or $x = y$.
- There are more related concepts, e. g.
  - (total) preorder: (connex), reflexive, transitive
  - well-order: total order over S such that every non-empty subset has a least element