

Discrete Mathematics in Computer Science

B2. Countable Sets

Malte Helmert, Gabriele Röger

University of Basel

Discrete Mathematics in Computer Science

— B2. Countable Sets

B2.1 Cardinality of Infinite Sets

B2.2 Hilbert's Hotel

B2.3 \aleph_0 and Countable Sets

B2.1 Cardinality of Infinite Sets

Finite Sets Revisited

We already know:

- ▶ The **cardinality** $|S|$ measures the size of set S .
- ▶ A set is **finite** if it has a finite number of elements.
- ▶ The **cardinality** of a finite set is the **number of elements** it contains.
- ▶ For a finite set S , it holds that $|\mathcal{P}(S)| = 2^{|S|}$.

A set is **infinite** if it has an infinite number of elements.

- ▶ Do all infinite sets have the same cardinality?
- ▶ Does the power set of infinite set S have the same cardinality as S ?

Comparing the Cardinality of Sets

- ▶ $\{1, 2, 3\}$ and $\{\text{dog, cat, mouse}\}$ have cardinality 3.
- ▶ We can pair their elements:

$1 \leftrightarrow \text{dog}$

$2 \leftrightarrow \text{cat}$

$3 \leftrightarrow \text{mouse}$

- ▶ We call such a mapping a **bijection** from one set to the other.
 - ▶ Each element of one set is paired with exactly one element of the other set.
 - ▶ Each element of the other set is paired with exactly one element of the first set.

Equinumerous Sets

We use the existence of a pairing also as criterion for infinite sets:

Definition (Equinumerous Sets)

Two sets A and B have the same cardinality ($|A| = |B|$) if there **exists a bijection from A to B** .

Such sets are called **equinumerous**.

When is a set “smaller” than another set?

Comparing the Cardinality of Sets

- ▶ Consider $A = \{1, 2\}$ and $B = \{\text{dog, cat, mouse}\}$.
- ▶ We can map distinct elements of A to distinct elements of B :

$1 \mapsto \text{dog}$

$2 \mapsto \text{cat}$

- ▶ We call this an **injective function** from A to B :
 - ▶ every element of A is mapped to an element of B ;
 - ▶ different elements of A are mapped to different elements of B .

Comparing Cardinality

Definition (cardinality not larger)

Set A has **cardinality less than or equal** to the cardinality of set B ($|A| \leq |B|$), if **there is an injective function from A to B** .

Definition (strictly smaller cardinality)

Set A has **cardinality strictly less** than the cardinality of set B ($|A| < |B|$), if $|A| \leq |B|$ and $|A| \neq |B|$.

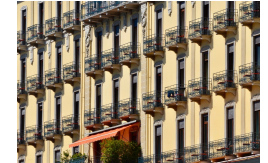
Consider set A and object $e \notin A$. Is $|A| < |A \cup \{e\}|$?

B2.2 Hilbert's Hotel

Hilbert's Hotel

Our intuition for finite sets does not always work for infinite sets.

- ▶ If in a hotel all rooms are occupied then it cannot accommodate additional guests.
- ▶ But **Hilbert's Grand Hotel** has **infinitely many rooms**.
- ▶ All these rooms are **occupied**.

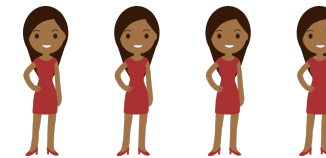


One More Guest Arrives



- ▶ Every guest moves from her current room n to room $n + 1$.
- ▶ Room 1 is then free.
- ▶ The new guest gets room 1.

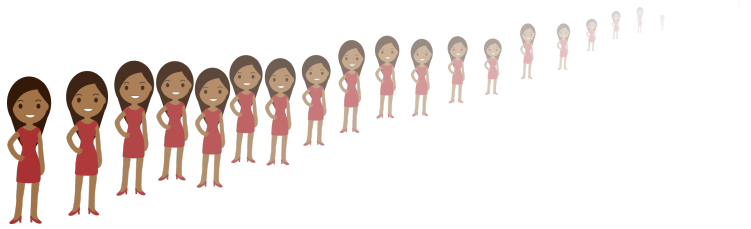
Four More Guests Arrive



- ▶ Every guest moves from her current room n to room $n + 4$.
- ▶ Rooms 1 to 4 are no longer occupied and can be used for the new guests.

→ Works for any finite number of additional guests.

An Infinite Number of Guests Arrives



- ▶ Every guest moves from her current room n to room $2n$.
- ▶ The infinitely many rooms with odd numbers are now available.
- ▶ The new guests fit into these rooms.

Can we Go further?

What if ...

- ▶ infinitely many coaches, each with an infinite number of guests
 - ▶ infinitely many ferries, each with an infinite number of coaches, each with infinitely many guests
 - ▶ ...
- ... arrive?

There are strategies for all these situations as long as with “infinite” we mean “countably infinite” and there is a finite number of layers.

B2.3 \aleph_0 and Countable Sets

Comparing Cardinality

- ▶ Two sets A and B have the **same cardinality** if their elements can be paired (i.e. there is a bijection from A to B).
- ▶ Set A has a **strictly smaller cardinality** than set B if
 - ▶ we can map distinct elements of A to distinct elements of B (i.e. there is an injective function from A to B), and
 - ▶ $|A| \neq |B|$.
- ▶ This clearly makes sense for finite sets.
- ▶ What about infinite sets?
Do they even have different cardinalities?

The Cardinality of the Natural Numbers

Definition (\aleph_0)

The **cardinality** of \mathbb{N}_0 is denoted by \aleph_0 , i.e. $\aleph_0 = |\mathbb{N}_0|$

Read: “aleph-zero”, “aleph-nought” or “aleph-null”

Countable and Countably Infinite Sets

Definition (countably infinite and countable)

A set A is **countably infinite** if $|A| = |\mathbb{N}_0|$.

A set A is **countable** if $|A| \leq |\mathbb{N}_0|$.

A set is **countable** if it is **finite** or **countably infinite**.

- ▶ We can count the elements of a countable set one at a time.
- ▶ The objects are “discrete” (in contrast to “continuous”).
- ▶ Discrete mathematics deals with all kinds of countable sets.

Set of Even Numbers

- ▶ $even = \{n \mid n \in \mathbb{N}_0 \text{ and } n \text{ is even}\}$
- ▶ Obviously: $even \subset \mathbb{N}_0$
- ▶ Intuitively, there are twice as many natural numbers as even numbers — no?
- ▶ Is $|even| < |\mathbb{N}_0|$?

Set of Even Numbers

Theorem (set of even numbers is countably infinite)

The set of all **even natural numbers** is **countably infinite**, i.e. $|\{n \mid n \in \mathbb{N}_0 \text{ and } n \text{ is even}\}| = |\mathbb{N}_0|$.

Proof Sketch.

We can pair every natural number n with the even number $2n$. \square

Set of Perfect Squares

Theorem (set of perfect squares is countably infinite)

The set of all **perfect squares** is **countably infinite**,
i. e. $|\{n^2 \mid n \in \mathbb{N}_0\}| = |\mathbb{N}_0|$.

Proof Sketch.

We can pair every natural number n with square number n^2 . \square

Subsets of Countable Sets are Countable

In general:

Theorem (subsets of countable sets are countable)

Let A be a countable set. Every set B with $B \subseteq A$ is countable.

Proof.

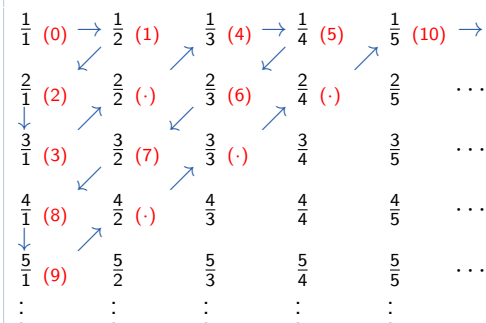
Since A is countable there is an injective function f from A to \mathbb{N}_0 .
The restriction of f to B is an injective function from B to \mathbb{N}_0 . \square

Set of the Positive Rationals

Theorem (set of positive rationals is countably infinite)

Set $\mathbb{Q}_+ = \{n \mid n \in \mathbb{Q} \text{ and } n > 0\} = \{p/q \mid p, q \in \mathbb{N}_1\}$
is **countably infinite**.

Proof idea.



\square

Union of Two Countable Sets is Countable

Theorem (union of two countable sets countable)

Let A and B be countable sets. Then $A \cup B$ is countable.

Proof sketch.

As A and B are countable there is an injective function f_A from A to \mathbb{N}_0 , analogously f_B from B to \mathbb{N}_0 .

We define function $f_{A \cup B}$ from $A \cup B$ to \mathbb{N}_0 as

$$f_{A \cup B}(e) = \begin{cases} 2f_A(e) & \text{if } e \in A \\ 2f_B(e) + 1 & \text{otherwise} \end{cases}$$

This $f_{A \cup B}$ is an injective function from $A \cup B$ to \mathbb{N}_0 . \square

Integers and Rationals

Theorem (sets of integers and rationals are countably infinite)

The sets \mathbb{Z} and \mathbb{Q} are *countably infinite*.

Without proof (\rightsquigarrow exercises)

Union of More than Two Sets

Definition (arbitrary unions)

Let M be a set of sets. The union $\bigcup_{S \in M} S$ is the set with

$$x \in \bigcup_{S \in M} S \text{ iff exists } S \in M \text{ with } x \in S.$$

Countable Union of Countable Sets

Theorem

Let M be a *countable set of countable sets*.

Then $\bigcup_{S \in M} S$ is *countable*.

We prove this formally after we have studied functions.

Set of all Binary Trees is Countable

Theorem (set of all binary trees is countable)

The set $B = \{b \mid b \text{ is a binary tree}\}$ is *countable*.

Proof.

For $n \in \mathbb{N}_0$ the set B_n of all binary trees with n leaves is finite.

With $M = \{B_i \mid i \in \mathbb{N}_0\}$ the set of all binary trees is

$$B = \bigcup_{B' \in M} B'.$$

Since M is a countable set of countable sets, B is countable. \square

And Now?

We have seen several sets with cardinality \aleph_0 .

What about our original questions?

- ▶ Do all infinite sets have the same cardinality?
- ▶ Does the power set of infinite set S have the same cardinality as S ?