# Discrete Mathematics in Computer Science

## Sets

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# Important Building Blocks of Discrete Mathematics

- sets
- relations
- functions

# Sets

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- unorderd: no notion of a “first” or “second” object, e. g. $\{Alice, Bob, Charly\} = \{Charly, Bob, Alice\}$
- distinct: each object contained at most once, e. g. $\{Alice, Bob, Charly\} = \{Alice, Charly, Bob, Alice\}$

# Notation

- Specification of sets
  - explicit, listing all elements, e. g. $A = \{1, 2, 3\}$
  - implicit with set-builder notation,
    specifying a property characterizing all elements,
    e. g. $A = \{x \mid x \in \mathbb{N}_0 \text{ and } 1 \leq x \leq 3\}$,
    $B = \{n^2 \mid n \in \mathbb{N}_0\}$
  - implicit, as a sequence with dots,
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Question: Is it true that $1 \in \{\{1, 2\}, 3\}$?

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- Rational numbers $\mathbb{Q} = \{n/d \mid n \in \mathbb{Z}, d \in \mathbb{N}_1\}$
- Real numbers $\mathbb{R} = (-\infty, \infty)$
  Why do we use interval notation?
  Why didn't we introduce it before?

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## Russell's Paradox

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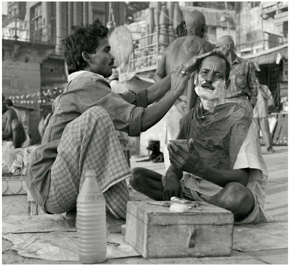
# Excursus: Barber Paradox

**Barber Paradox**

In a town there is only one barber,
who is male.

The barber shaves all men in the town,
and only those,
who do not shave themselves.



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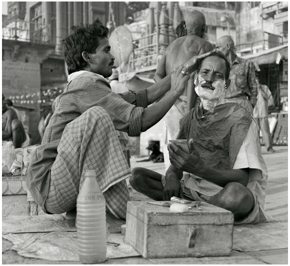

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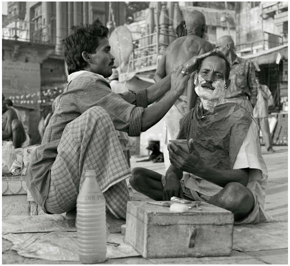

We can exploit the self-reference to derive a contradiction.

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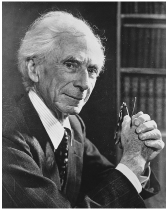

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Is the collection of all sets
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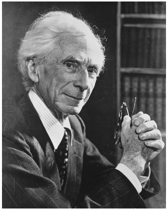

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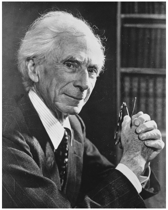

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Is $S = \{M \mid M \text{ is a set and } M \notin M\}$ a set?

Assume that $S$ is a set.
If $S \notin S$ then $S \in S \rightsquigarrow$ Contradiction
If $S \in S$ then $S \notin S \rightsquigarrow$ Contradiction
Hence, there is no such set $S$.

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## Relations on Sets

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# Equality

**Definition (Axiom of Extensionality)**

Two sets $A$ and $B$ are equal (written $A = B$)
if every element of $A$ is an element of $B$ and vice versa.

Two sets are equal if they contain the same elements.

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Two sets are equal if they contain the same elements.

We write $A \neq B$ to indicate that $A$ and $B$ are not equal.

# Subsets and Supersets

- $A \subseteq B$: $A$ is a subset of $B$,
  i. e., every element of $A$ is an element of $B$
- $A \subset B$: $A$ is a strict subset of $B$,
  i. e., $A \subseteq B$ and $A \neq B$.
- $A \supseteq B$: $A$ is a superset of $B$ if $B \subseteq A$.
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We write $A \not\subseteq B$ to indicate that $A$ is not a subset of $B$.

Analogously: $\not\subset$, $\not\supseteq$, $\not\supset$

# Power Set

**Definition (Power Set)**

The power set $\mathcal{P}(S)$ of a set $S$ is the set of all subsets of $S$.
That is,

$$\mathcal{P}(S) = \{M \mid M \subseteq S\}.$$

Example: $\mathcal{P}(\{a, b\}) =$

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## Set Operations

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Set operations allow us to express sets in terms of other sets

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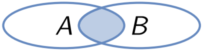
- intersection $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$




If $A \cap B = \emptyset$ then $A$ and $B$ are disjoint.

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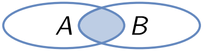


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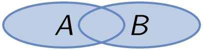

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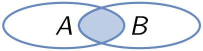


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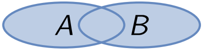
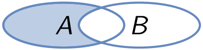



- set difference $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$



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  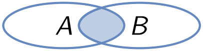
  

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  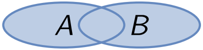
  

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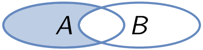
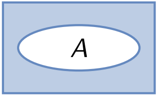
  
  

- complement $\overline{A} = B \setminus A$, where $A \subseteq B$ and
  $B$ is the set of all considered objects (in a given context)

# Properties of Set Operations: Commutativity

**Theorem (Commutativity of $\cup$ and $\cap$)**

*For all sets $A$ and $B$ it holds that*

- $A \cup B = B \cup A$ *and*
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Question: Is the set difference also commutative,
i. e. is $A \setminus B = B \setminus A$ for all sets $A$ and $B$?

# Properties of Set Operations: Associativity

**Theorem (Associativity of $\cup$ and $\cap$)**

*For all sets $A, B$ and $C$ it holds that*

- $(A \cup B) \cup C = A \cup (B \cup C)$ *and*
- $(A \cap B) \cap C = A \cap (B \cap C)$.

# Properties of Set Operations: Distributivity

**Theorem (Union distributes over intersection and vice versa)**

*For all sets $A, B$ and $C$ it holds that*

- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ *and*
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

# Properties of Set Operations: De Morgan's Law

Augustus De Morgan

British mathematician (1806-1871)

**Theorem (De Morgan's Law)**

*For all sets $A$ and $B$ it holds that*

- $\overline{A \cup B} = \overline{A} \cap \overline{B}$ *and*
- $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

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## Finite Sets

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# Cardinality of Sets

The cardinality $|S|$ measures the size of set $S$.

A set is finite if it has a finite number of elements.

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The cardinality of a finite set is the number of elements it contains.

- $|\emptyset| =$
- $|\{x \mid x \in \mathbb{N}_0 \text{ and } 2 \leq x < 5\}| =$
- $|\{3, 0, \{1, 3\}\}| =$

# Cardinality of the Union of Sets

**Theorem**

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*For finite sets $A$ and $B$ it holds that* $|A \cup B| = |A| + |B| - |A \cap B|$.

**Corollary**

*If finite sets $A$ and $B$ are disjoint then* $|A \cup B| = |A| + |B|$.

# Cardinality of the Power Set

**Theorem**

*Let $S$ be a finite set. Then $|\mathcal{P}(S)| = 2^{|S|}$.*

**Proof sketch.**

We can construct a subset $S'$ by iterating over all elements $e$ of $S$ and deciding whether $e$ becomes a member of $S'$ or not.

We make $|S|$ independent decisions, each between two options. Hence, there are $2^{|S|}$ possible outcomes.

Every subset of $S$ can be constructed this way and different choices lead to different sets. Thus, $|\mathcal{P}(S)| = 2^{|S|}$. □

# Alternative Proof by Induction

**Proof.**

By induction over $|S|$.

Basis ($|S| = 0$): Then $S = \emptyset$ and $|\mathcal{P}(S)| = |\{\emptyset\}| = 1 = 2^0$.

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Inductive Step ($n \to n + 1$):

Let $S'$ be an arbitrary set with $|S'| = n + 1$ and
let $e$ be an arbitrary member of $S'$.

Let further $S = S' \setminus \{e\}$ and $X = \{S'' \cup \{e\} \mid S'' \in \mathcal{P}(S)\}$.

Then $\mathcal{P}(S') = \mathcal{P}(S) \cup X$. As $\mathcal{P}(S)$ and $X$ are disjoint and $|X| = |\mathcal{P}(S)|$, it holds that $|\mathcal{P}(S')| = 2|\mathcal{P}(S)|$.

Since $|S| = n$, we can use the IH and get

$$|\mathcal{P}(S')| = 2 \cdot 2^{|S|} = 2 \cdot 2^n = 2^{n+1} = 2^{|S'|}.$$

□

# Enumerating all Subsets

Determine a one-to-one mapping between numbers $0, \ldots, 2^{|S|} - 1$ and all subsets of finite set $S$:

- Consider the binary representation of numbers $0, \ldots, 2^{|S|} - 1$.
- Associate every bit with a different element of $S$.
- Every number is mapped to the set that contains exactly the elements associated with the 1-bits.

$S = \{a, b, c\}$

| decimal | binary<br>cba | set |
|---|---|---|
| 0 | 000 | $\{\}$ |
| 1 | 001 | $\{a\}$ |
| 2 | 010 | $\{b\}$ |
| 3 | 011 | $\{a, b\}$ |
| 4 | 100 | $\{c\}$ |
| 5 | 101 | $\{a, c\}$ |
| 6 | 110 | $\{b, c\}$ |
| 7 | 111 | $\{a, b, c\}$ |

# Computer Representation as Bit String

Same representation as in enumeration of all subsets:

- Required: Fixed universe $U$ of possible elements
- Represent sets as bitstrings of length $|U|$
- Associate every bit with one object from the universe
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Example:

- $U = \{o_0, \ldots, o_9\}$
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How can the set operations be implemented?