Discrete Mathematics in Computer Science

Mathematical Induction

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Proof Techniques

most common proof techniques:

- direct proof
- indirect proof (proof by contradiction)
- contrapositive
- mathematical induction
- structural induction

Mathematical Induction

Concrete Mathematics by Graham, Knuth and Patashnik (p. 3)

Mathematical induction proves that

we can climb as high as we like on a ladder,

by proving that we can climb onto the bottom rung (the basis)

and that

from each rung we can climb up to the next one (the step).

Propositions

Consider a statement on all natural numbers n with $n \geq m$.

- E.g. "Every natural number $n \geq 2$ can be written as a product of prime numbers."
  - $P(2)$: "2 can be written as a product of prime numbers."
  - $P(3)$: "3 can be written as a product of prime numbers."
  - $P(4)$: "4 can be written as a product of prime numbers."
  - ...
  - $P(n)$: "n can be written as a product of prime numbers."
  - For every natural number $n \geq 2$ proposition $P(n)$ is true.

A proposition $P(n)$ is a mathematical statement that is defined in terms of natural number n.

Mathematical Induction

Mathematical Induction

Proof (of the truth) of proposition $P(n)$
for all natural numbers n with $n \geq m$:

- basis: proof of $P(m)$
- induction hypothesis (IH):
  suppose that $P(k)$ is true for all k with $m \leq k \leq n$
- inductive step: proof of $P(n+1)$
  using the induction hypothesis

Mathematical Induction: Example I

Theorem

For all $n \in \mathbb{N}_0$ *with* $n \geq 1$*:* $\sum_{k=1}^{n}(2k-1) = n^2$

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Proof.

Mathematical induction over n:

basis $n = 1$: $\sum_{k=1}^{1}(2k-1) = 2 - 1 = 1 = 1^2$

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inductive step $n \to n+1$:

$$\begin{aligned}\sum_{k=1}^{n+1}(2k-1) &= \Big(\sum_{k=1}^{n}(2k-1)\Big) + 2(n+1) - 1 \\ &\overset{\text{IH}}{=} n^2 + 2(n+1) - 1 \\ &= n^2 + 2n + 1 = (n+1)^2\end{aligned}$$

□

Mathematical Induction: Example II

Theorem

Every natural number $n \geq 2$ can be written as a product of prime numbers, i. e. $n = p_1 \cdot p_2 \cdot \ldots \cdot p_m$ with prime numbers $p_1, \ldots, p_m$.

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IH: Every natural number k with $2 \leq k \leq n$
can be written as a product of prime numbers. . . .

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- Case 1: $n + 1$ is a prime number $\rightsquigarrow$ trivial

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inductive step $n \to n + 1$:

- Case 1: $n + 1$ is a prime number $\rightsquigarrow$ trivial
- Case 2: $n + 1$ is not a prime number.
  There are natural numbers $2 \leq q, r \leq n$ with $n + 1 = q \cdot r$.
  Using IH shows that there are prime numbers
  $q_1, \ldots, q_s$ with $q = q_1 \cdot \ldots \cdot q_s$ and
  $r_1, \ldots, r_t$ with $r = r_1 \cdot \ldots \cdot r_t$.
  Together this means $n + 1 = q_1 \cdot \ldots \cdot q_s \cdot r_1 \cdot \ldots \cdot r_t$.

□

Weak vs. Strong Induction

- Weak induction: Induction hypothesis only supposes that $P(k)$ is true for $k = n$
- Strong induction: Induction hypothesis supposes that $P(k)$ is true for all $k \in \mathbb{N}_0$ with $m \leq k \leq n$
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Our previous definition corresponds to strong induction.

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Which of the examples had also worked with weak induction?

Is Strong Induction More Powerful than Weak Induction?

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Are there statements that we can prove with strong induction but not with weak induction?

We can always use a stronger proposition:

- "Every $n \in \mathbb{N}_0$ with $n \geq 2$ can be written as a product of prime numbers."
- $P(n)$: "n can be written as a product of prime numbers."
- $P'(n)$: "all $k \in \mathbb{N}_0$ with $2 \leq k \leq n$ can be written as a product of prime numbers."

Reformulating Statements

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Be careful about how to reformulate a statement!

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Structural Induction

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Inductively Defined Sets: Examples

Example (Natural Numbers)

The set $\mathbb{N}_0$ of natural numbers is inductively defined as follows:

- 0 is a natural number.
- If n is a natural number, then $n + 1$ is a natural number.

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Example (Binary Tree)

The set $\mathcal{B}$ of binary trees is inductively defined as follows:

- $\square$ is a binary tree (a leaf)
- If L and R are binary trees, then $\langle L, \bigcirc, R \rangle$ is a binary tree (with inner node $\bigcirc$).

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Implicit statement: all elements of the set can be constructed by finite application of these rules

Inductive Definition of a Set

Inductive Definition

A set M can be defined inductively by specifying

- basic elements that are contained in M
- construction rules of the form
  "Given some elements of M, another element of M can be constructed like this."

Structural Induction

Structural Induction

Proof of statement for all elements of an inductively defined set

- basis: proof of the statement for the basic elements
- induction hypothesis (IH):
  suppose that the statement is true for some elements M
- inductive step: proof of the statement for elements constructed by applying a construction rule to M
  (one inductive step for each construction rule)

Structural Induction: Example (1)

Definition (Leaves of a Binary Tree)

The number of leaves of a binary tree B, written $\mathit{leaves}(B)$, is defined as follows:

$$\mathit{leaves}(\square) = 1$$
$$\mathit{leaves}(\langle L, \bigcirc, R\rangle) = \mathit{leaves}(L) + \mathit{leaves}(R)$$

Definition (Inner Nodes of a Binary Tree)

The number of inner nodes of a binary tree B, written $\mathit{inner}(B)$, is defined as follows:

$$\mathit{inner}(\square) = 0$$
$$\mathit{inner}(\langle L, \bigcirc, R\rangle) = \mathit{inner}(L) + \mathit{inner}(R) + 1$$

Structural Induction: Example (2)

Theorem

For all binary trees B: $inner(B) = leaves(B) - 1$.

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Theorem

For all binary trees B: $\mathit{inner}(B) = \mathit{leaves}(B) - 1$.

Proof.

induction basis:

$\mathit{inner}(\Box) = 0 = 1 - 1 = \mathit{leaves}(\Box) - 1$

$\rightsquigarrow$ statement is true for base case ...

Structural Induction: Example (3)

Proof (continued).

induction hypothesis:

to prove that the statement is true for a composite tree $\langle L, \bigcirc, R \rangle$, we may use that it is true for the subtrees L and R.

□

Structural Induction: Example (3)

Proof (continued).

induction hypothesis:

to prove that the statement is true for a composite tree $\langle L, \bigcirc, R\rangle$, we may use that it is true for the subtrees L and R.

inductive step for $B = \langle L, \bigcirc, R\rangle$:

$$\begin{aligned} \mathit{inner}(B) &= \mathit{inner}(L) + \mathit{inner}(R) + 1 \\ &\overset{\text{IH}}{=} (\mathit{leaves}(L) - 1) + (\mathit{leaves}(R) - 1) + 1 \\ &= \mathit{leaves}(L) + \mathit{leaves}(R) - 1 = \mathit{leaves}(B) - 1 \end{aligned}$$

□

Structural Induction: Exercise

Definition (Height of a Binary Tree)

The height of a binary tree B, written $\mathit{height}(B)$,
is defined as follows:

$$\mathit{height}(\square) = 0$$
$$\mathit{height}(\langle L, \bigcirc, R\rangle) = \max\{\mathit{height}(L), \mathit{height}(R)\} + 1$$

Prove by structural induction:

Theorem

For all binary trees B: $\mathit{leaves}(B) \leq 2^{\mathit{height}(B)}$.