Discrete Mathematics in Computer Science

A2. Proofs I

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A2.1 What is a Proof?

What is a Proof?

A mathematical proof is

- a sequence of logical steps
- starting with one set of statements
- that comes to the conlusion
  that some statement must be true.

What is a statement?

Mathematical Statements

Mathematical Statement

A mathematical statement consists of a set of preconditions and a set of conclusions.

The statement is true if the conclusions are true whenever the preconditions are true.

Notes:

- set of preconditions is sometimes empty
- often, "assumptions" is used instead of "preconditions"; slightly unfortunate because "assumption" is also used with another meaning ($\rightsquigarrow$ cf. indirect proofs)

Examples of Mathematical Statements

Examples (some true, some false):

- "Let $p \in \mathbb{N}_0$ be a prime number. Then p is odd."
- "There exists an even prime number."
- "Let $p \in \mathbb{N}_0$ with $p \geq 3$ be a prime number. Then p is odd."
- "All prime numbers $p \geq 3$ are odd."
- "For all sets A, B, C: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$"
- "0 is a natural number."
- "The equation $a^k + b^k = c^k$ has infinitely many solutions with $a, b, c, k \in \mathbb{N}_1$ and $k \geq 2$."
- "The equation $a^k + b^k = c^k$ has no solutions with $a, b, c, k \in \mathbb{N}_1$ and $k \geq 3$."

What are the preconditions, what are the conclusions?

On what Statements can we Build the Proof?

A mathematical proof is

- a sequence of logical steps
- starting with one set of statements
- that comes to the conlusion
  that some statement must be true.

We can use:

- axioms: statements that are assumed to always be true
  in the current context
- theorems and lemmas: statements that were already proven
  - lemma: an intermediate tool
  - theorem: itself a relevant result
- premises: assumptions we make
  to see what consequences they have

What is a Logical Step?

A mathematical proof is

- a sequence of logical steps
- starting with one set of statements
- that comes to the conlusion
  that some statement must be true.

Each step directly follows

- from the axioms,
- premises,
- previously proven statements and
- the preconditions of the statement we want to prove.

For a formal definition, we would need formal logics.

The Role of Definitions

Definition

A set is an unordered collection of distinct objects.
The set that does not contain any objects is the *empty set* $\emptyset$.

- A definition introduces an abbreviation.
- Whenever we say "set", we could instead say "an unordered collection of distinct objects" and vice versa.
- Definitions can also introduce notation.

Disproofs

- A disproof (refutation) shows that a given mathematical statement is false by giving an example where the preconditions are true, but the conclusion is false.
- This requires deriving, in a sequence of proof steps, the opposite (negation) of the conclusion.

- Formally, disproofs are proofs of modified ("negated") statements.
- Be careful about how to negate a statement!

A Word on Style

A proof should help the reader to see why the result must be true.

- A proof should be easy to follow.
- Omit unnecessary information.
- Move self-contained parts into separate lemmas.
- In complicated proofs, reveal the overall structure in advance.
- Have a clear line of argument.

$\rightarrow$ Writing a proof is like writing an essay.

A2.2 Proof Strategies

Common Forms of Statements

Many statements have one of these forms:

1. "All $x \in S$ with the property P also have the property Q."
2. "A is a subset of B."
3. "For all $x \in S$: x has property P iff x has property Q."
4. "$A = B$", where A and B are sets.

In the following, we will discuss some typical proof/disproof strategies for such statements.

Proof Strategies

1. "All $x \in S$ with the property P also have the property Q."
   "For all $x \in S$: if x has property P, then x has property Q."
   - To prove, assume you are given an arbitrary $x \in S$ that has the property P.
     Give a sequence of proof steps showing that x must have the property Q.
   - To disprove, find a counterexample, i. e., find an $x \in S$ that has property P but not Q and prove this.

Proof Strategies

2. "A is a subset of B."
   - To prove, assume you have an arbitrary element $x \in A$ and prove that $x \in B$.
   - To disprove, find an element in $x \in A \setminus B$ and prove that $x \in A \setminus B$.

Proof Strategies

3. "For all $x \in S$: x has property P iff x has property Q."
   ("iff": "if and only if")
   - To prove, separately prove "if P then Q" and "if Q then P".
   - To disprove, disprove "if P then Q" or disprove "if Q then P".

Proof Strategies

4. "$A = B$", where A and B are sets.
   - To prove, separately prove "$A \subseteq B$" and "$B \subseteq A$".
   - To disprove, disprove "$A \subseteq B$" or disprove "$B \subseteq A$".

Proof Techniques

most common proof techniques:

- direct proof
- indirect proof (proof by contradiction)
- contrapositive
- mathematical induction
- structural induction

A2.3 Direct Proof

Direct Proof

Direct Proof

Direct derivation of the statement by deducing or rewriting.

Direct Proof: Example

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A2.4 Indirect Proof

Indirect Proof

Indirect Proof (Proof by Contradiction)

- Make an assumption that the statement is false.
- Derive a contradiction from the assumption together with the preconditions of the statement.
- This shows that the assumption must be false given the preconditions of the statement, and hence the original statement must be true.

Indirect Proof: Example

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A2.5 Proof by Contrapositive

Contrapositive

(Proof by) Contrapositive

Prove "If A, then B" by proving "If not B, then not A."

Examples:

- Prove "For all $n \in \mathbb{N}_0$: if n^2 is odd, then n is odd" by proving "For all $n \in \mathbb{N}_0$, if n is even, then n^2 is even."
- Prove "For all $n \in \mathbb{N}_0$: if n is not a square number, then $\sqrt{n}$ is irrational" by proving "For all $n \in \mathbb{N}_0$: if $\sqrt{n}$ is rational, then n is a square number."

Contrapositive: Example

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A2.6 Excursus: Computer-assisted Theorem Proving

Computer-assisted Proofs

- Computers can help proving theorems.
- Computer-aided proofs have for example been used for proving theorems by exhaustion.
- Example: Four color theorem

Interactive Theorem Proving

- On the lowest abstraction level, rigorous mathematical proofs rely on formal logic.
- On this level, proofs can be automatically verified by computers.
- Nobody wants to write or read proofs on this level of detail.
- In Interactive Theorem Proving a human guides the proof and the computer tries to fill in the details.
- If it succeeds, we can be very confident that the proof is valid.
- Example theorem provers: Isabelle/HOL, Lean