Discrete Mathematics in Computer Science

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Exercise Sheet 12

Due: Thursday, December 16, 2021

Exercise 12.1 (1 mark)

Specify *all* reasons why $\Big(\forall\exists xy\big((\mathrm{f}(x) = \mathrm{g}(y)) \wedge \mathrm{P}(\neg x)\big) \vee (\mathrm{Q}(x, \mathrm{c}) = c)\Big)$ is not a predicate logic formula over signature $\langle\{x, y\}, \{\mathrm{c}\}, \{\mathrm{f}, \mathrm{g}\}, \{\mathrm{P}, \mathrm{Q}\}\rangle$ with $ar(\mathrm{f}) = ar(\mathrm{P}) = 1$ and $ar(\mathrm{g}) = ar(\mathrm{Q}) = 2$.

Exercise 12.2 (2 marks)

Formalize the following statements about natural numbers as predicate logic formulas over signature $\langle\{x, y\}, \{\mathrm{zero}, \mathrm{one}\}, \{\mathrm{sum}, \mathrm{product}\}, \{\mathrm{Odd}, \mathrm{Even}\}\rangle$ with $ar(\mathrm{sum}) = ar(\mathrm{product}) = 2$ and $ar(\mathrm{Odd}) = ar(\mathrm{Even}) = 1$.
Note: When formalizing statements we do not care whether the statement is true under the usual interpretation.

(a) There are no two consecutive numbers whose product is odd.

(b) The square of an even number is always odd.

Exercise 12.3 (3 marks)

Consider the following predicate logic formula φ over signature $\langle\{x, y\}, \{\mathrm{c}\}, \{\mathrm{f}, \mathrm{g}\}, \{\mathrm{P}\}\rangle$:

$$\varphi = \Big(\neg\mathrm{P}(\mathrm{c}) \wedge \forall x\exists y\,\big((\mathrm{f}(y) = \mathrm{g}(x)) \wedge \mathrm{P}(y)\big)\Big)$$

Specify a model $\mathcal{I}$ of φ with $\mathcal{I} = \langle U, \cdot^{\mathcal{I}}\rangle$ and $\mathcal{U} = \{u_1, u_2, u_3\}$, and *prove* that $\mathcal{I} \models \varphi$ holds. Why is no variable assignment α required to specify a model of φ?
Hint: Note that the majority of points is given to the proof, so pay attention to writing a clean proof.

Exercise 12.4 (1 mark)

Consider formula φ over signature $\langle\{x, y, z\}, \{\mathrm{c}\}, \{\mathrm{f}\}, \{\mathrm{P}, \mathrm{Q}, \mathrm{R}\}\rangle$ with $ar(\mathrm{f}) = ar(\mathrm{P}) = 1$, $ar(\mathrm{Q}) = 2$ and $ar(\mathrm{R}) = 3$.

$$\varphi = \Big(\forall x\exists y\,\big(\mathrm{P}(z) \rightarrow \mathrm{Q}(y, x)\big) \vee \neg\exists y\,\mathrm{R}(\mathrm{c}, x, \mathrm{f}(y))\Big)$$

Mark all occurrences of free variables in φ. *Additionally* specify the set of free variables of φ (without proof).

Exercise 12.5 (1 mark)

Is $\varphi = (\mathrm{sum}(\mathrm{zero}, \mathrm{zero}) = \mathrm{zero})$ over signature $\langle\emptyset, \{\mathrm{zero}\}, \{\mathrm{sum}\}, \emptyset\rangle$ *with* $ar(\mathrm{sum}) = 2$ valid? If not, specify a counterexample. Briefly justify your answer.

Exercise 12.6 (2 marks)

Refute the following statement: For all formulas φ and ψ the statement $(\exists x\varphi \wedge \exists x\psi) \models \exists x(\varphi \wedge \psi)$ holds.

Submission rules:
Upload a single PDF file (ending .pdf) generated using LaTeX. Put the names of all group members on top of the first page. Use page numbers or put your names on each page. Make sure your PDF has size A4 (fits the page size if printed on A4).