Discrete Mathematics in Computer Science

M. Helmert, G. Röger
S. Eriksson
Fall Term 2021

University of Basel
Computer Science

Exercise Sheet 2

Due: Thursday, October 7, 2021

Exercise 2.1 (1 mark)

We want to prove that all toucans have equally long beaks. Which line of the following proof by mathematical induction has an error? What exactly is the error on the formal level?

1. Since we do not know the exact number of toucans we will prove that for any set of toucans all toucans in the set have equally long beaks.
2. We define the property $T(n)$: In any set of n toucans, all toucans have equally long beaks.
3. As the induction basis we show $T(1)$.
4. In a set of toucans consisting of only one toucan it is obvious that all toucans in the set have equally long beaks. Thus $T(1)$ holds.
5. The induction hypothesis is: $T(i)$ holds for all $1 \leq i \leq n$.
6. We need to show $T(n+1)$ under the assumption of the induction hypothesis.
7. We consider a set of $n+1$ toucans $M = \{t_1, \dots, t_n, t_{n+1}\}$
8. Removing the first toucan from the set M results in a set $M_1 = \{t_2, \dots, t_{n+1}\}$ containing n toucans ($|M_1| = n$).
9. According to the induction hypothesis $T(n)$ holds for M_1 and thus we can conclude that all toucans in M_1 have equally long beaks.
10. Removing the last toucan from the set M results in a set $M_2 = \{t_1, \dots, t_n\}$ containing n toucans ($|M_2| = n$).
11. According to the induction hypothesis $T(n)$ holds for M_2 and thus we can conclude that all toucans in M_2 have equally long beaks.
12. Now we consider a toucan t_M which is contained in both sets, for example $t_M = t_2$.
13. t_M has a beak that is equally long as the ones of all toucans from M_1 and also as the ones of all toucans from M_2.
14. From this we conclude that all toucans from $M_1 \cup M_2 = M$ must have equally long beaks, which means we have shown that $T(n+1)$ holds for M.
15. Since $T(1)$ holds, and, under hypothesis $T(n)$, $T(n+1)$ also holds for $n \geq 1$, we conclude that $T(n)$ holds for all $n \in \mathbb{N}$.
16. In particular $T(n)$ holds for the set of all toucans, which means that all toucans have equally long beaks.

Exercise 2.2 (3 marks)

We consider the set $\mathcal{B}$ of binary trees which is inductively defined as follows:

- $\Box$ is a binary tree.
- If L and R are binary trees, then $\langle L, \bigcirc, R\rangle$ is a binary tree.

Given a binary tree B we define its number of leaves with

$$\begin{aligned} \mathit{leaves}(\Box) &= 1 \\ \mathit{leaves}(\langle L, \bigcirc, R\rangle) &= \mathit{leaves}(L) + \mathit{leaves}(R) \end{aligned}$$

,

and its number of edges with

$$\begin{aligned} \mathit{edges}(\Box) &= 0 \\ \mathit{edges}(\langle L, \bigcirc, R\rangle) &= \mathit{edges}(L) + \mathit{edges}(R) + 2 \end{aligned}$$

Show with structural induction that for all binary trees B we have $\mathit{edges}(B) = 2 \cdot \mathit{leaves}(B) - 2$.

Exercise 2.3 (1 mark)

The following set definitions are syntactically wrong. Specify the correct syntax.

(a) The set of all numbers that are squares or cubes of a natural number: $\{x^2, x^3 \mid x \in \mathbb{N}_0\}$

(b) The set consisting of all even numbers and the number 7: $\{2x \mid x \in \mathbb{N}_0\} \cup 7$

Exercise 2.4 (2 marks)

(a) Define set $S_1 = \{\{x, y\} \mid x \in \{1, 2, 3\}, y = 2\}$ as explicit enumeration.

(b) Define set $S_2 = \{3, 9, 15, 21, 27, 33, \dots\}$ with the set-builder notation.

(c) Define set $S_3 = \{2n \mid n \in \mathbb{N}_0\}$ inductively.

(d) Consider set S_4 which is inductively defined as follows:

- $\{0\} \in S_4$,
- if $\{x\} \in S_4$ then $\{x + 2\} \in S_4$ and
- if $S \in S_4$ and $S' \in S_4$ then $(S \cup S') \in S_4$.

Describe S_4 in natural language.

Exercise 2.5 (3 marks)

We consider finite sets $A, B \subseteq \mathbb{N}_0$. If A and B satisfy the following properties, how do they relate to each other? Briefly justify your answer.
Note: "briefly justify" means you do not need to write a formal proof, only a high-level argument for why your answer is correct.

(a) $|A \cap B| = |A|$

(b) $A \setminus B = B \setminus A$

(c) $\mathcal{P}(A) \cap \mathcal{P}(B) = \{\emptyset\}$

Submission rules:

Upload a single PDF file (ending .pdf) generated using LaTeX. Put the names of all group members on top of the first page. Use page numbers or put your names on each page. Make sure your PDF has size A4 (fits the page size if printed on A4).