Planning and Optimization

G1. Factored MDPs

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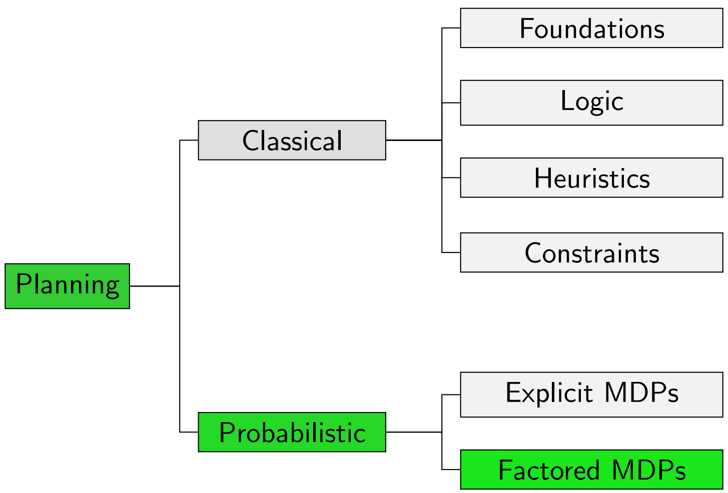
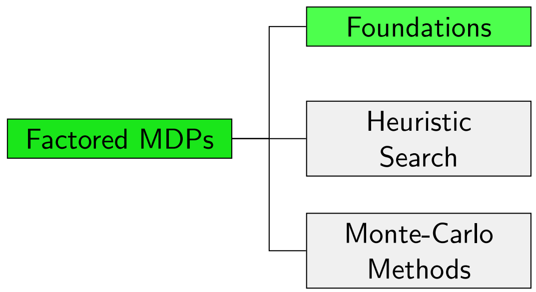
Content of this Course



Content of this Course: Factored MDPs



G1.1 Factored MDPs

Factored MDPs

We would like to specify MDPs and SSPs with large state spaces.
In classical planning, we introduced planning tasks to represent large transition systems compactly.

- represent aspects of the world in terms of state variables
- states are a valuation of state variables
- n (propositional) state variables induce 2^n states
  $\rightsquigarrow$ exponentially more compact than "explicit" representation

Finite-Domain State Variables

Definition (Finite-Domain State Variable)

A finite-domain state variable is a symbol v with an associated domain $\text{dom}(v)$, which is a finite non-empty set of values.

Let V be a finite set of finite-domain state variables.

A state s over V is an assignment $s : V \to \bigcup_{v \in V} \text{dom}(v)$ such that $s(v) \in \text{dom}(v)$ for all $v \in V$.

A formula over V is a propositional logic formula whose atomic propositions are of the form $v = d$ where $v \in V$ and $d \in \text{dom}(v)$.

For simplicity, we only consider finite-domain state variables here.

Syntax of Operators

Definition (SSP and MDP Operators)

An SSP operator o over a set of state variables V has three components:

- a precondition $pre(o)$, a logical formula over V
- an effect $\mathit{eff}(o)$ over V, defined on the following slides
- a cost $cost(o) \in \mathbb{R}^+_0$

An MDP operator o over a set of state variables V has three components:

- a precondition $pre(o)$, a logical formula over V
- an effect $\mathit{eff}(o)$ over V, defined on the following slides
- a reward $reward(o)$ over V, defined on the following slides

Whenever we just say operator (without SSP or MDP), both kinds of operators are allowed.

Syntax of Effects

Definition (Effect)

Effects over state variables V are inductively defined as follows:

- If $v \in V$ is a finite-domain state variable and $d \in \text{dom}(v)$, then $v := d$ is an effect (atomic effect).
- If $e_1, \dots, e_n$ are effects, then $(e_1 \wedge \dots \wedge e_n)$ is an effect (conjunctive effect).
  The special case with $n = 0$ is the empty effect $\top$.
- If $e_1, \dots, e_n$ are effects and $p_1, \dots, p_n \in [0, 1]$ such that $\sum_{i=1}^{n} p_i = 1$, then $(p_1 : e_1 | \dots | p_n : e_n)$ is an effect (probabilistic effect).

Note: To simplify definitions, conditional effects are omitted.

Effects: Intuition

Intuition for effects:

- Atomic effects can be understood as assignments that update the value of a state variable.
- A conjunctive effect $e = (e_1 \wedge \dots \wedge e_n)$ means that all subeffects $e_1, \dots, e_n$ take place simultaneously.
- A probabilistic effect $e = (p_1 : e_1 | \dots | p_n : e_n)$ means that exactly one subeffect $e_i \in \{e_1, \dots, e_n\}$ takes place with probability p_i.

Semantics of Effects

Definition

The effect set $[e]$ of an effect e is a set of pairs $\langle p, w \rangle$, where p is a probability $0 < p \leq 1$ and w is a partial assignment. The effect set $[e]$ is the set obtained recursively as

$$[v := d] = \{\langle 1.0, \{v \mapsto d\} \rangle\},$$

$$[e \wedge e'] = \biguplus_{\langle p, w \rangle \in [e], \langle p', w' \rangle \in [e']} \{\langle p \cdot p', w \cup w' \rangle\},$$

$$[p_1 : e_1 | \dots | p_n : e_n] = \biguplus_{i=1}^{n} \{\langle p_i \cdot p, w \rangle \mid \langle p, w \rangle \in [e_i]\}.$$

where $\uplus$ is like $\bigcup$ but merges $\langle p, w' \rangle$ and $\langle p', w' \rangle$ to $\langle p + p', w' \rangle$.

Semantics of Operators

Definition (Applicable, Outcomes)

Let V be a set of finite-domain state variables.
Let s be a state over V, and let o be an operator over V.
Operator o is applicable in s if $s \models pre(o)$.

The outcomes of applying an operator o in s, written $s[\![o]\!]$, are

$$s[\![o]\!] = \biguplus_{\langle p, w \rangle \in [eff(o)]} \{\langle p, s'_w \rangle\},$$

with $s'_w(v) = d$ if $v = d \in w$ and $s'_w(v) = s(v)$ otherwise and $\uplus$ is like $\bigcup$ but merges $\langle p, s' \rangle$ and $\langle p', s' \rangle$ to $\langle p + p', s' \rangle$.

Rewards

Definition (Reward)

A reward over state variables V is inductively defined as follows:

- $c \in \mathbb{R}$ is a reward
- If χ is a propositional formula over V, $[\chi]$ is a reward
- If r and r' are rewards, $r + r'$, $r - r'$, $r \cdot r'$ and $\frac{r}{r'}$ are rewards

Applying an MDP operator o in s induces reward $reward(o)(s)$, i.e., the value of the arithmetic function $reward(o)$ where all occurrences of $v \in V$ are replaced with $s(v)$.

G1.2 Probabilistic Planning Tasks

Probabilistic Planning Tasks

Definition (SSP and MDP Planning Task)

An SSP planning task is a 4-tuple $\Pi = \langle V, I, O, \gamma \rangle$ where

- V is a finite set of finite-domain state variables,
- I is a valuation over V called the initial state,
- O is a finite set of SSP operators over V, and
- γ is a formula over V called the goal.

An MDP planning task is a 4-tuple $\Pi = \langle V, I, O, d \rangle$ where

- V is a finite set of finite-domain state variables,
- I is a valuation over V called the initial state,
- O is a finite set of MDP operators over V, and
- $d \in (0, 1)$ is the discount factor.

A probabilistic planning task is an SSP or MDP planning task.

Mapping SSP Planning Tasks to SSPs

Definition (SSP Induced by an SSP Planning Task)

The SSP planning task $\Pi = \langle V, I, O, \gamma \rangle$ induces the SSP $\mathcal{T} = \langle S, A, c, T, s_0, S_\star \rangle$, where

- S is the set of all states over V,
- A is the set of operators O,
- $c(o) = cost(o)$ for all $o \in O$,
- $T(s, o, s') = \begin{cases} p & \text{if } o \text{ applicable in } s \text{ and } \langle p, s' \rangle \in s[\![o]\!] \\ 0 & \text{otherwise} \end{cases}$
- $s_0 = I$, and
- $S_\star = \{s \in S \mid s \models \gamma\}$.

Mapping MDP Planning Tasks to MDPs

Definition (MDP Induced by an MDP Planning Task)

The MDP planning task $\Pi = \langle V, I, O, d \rangle$ induces the MDP $\mathcal{T} = \langle S, A, R, T, s_0, \gamma \rangle$, where

- S is the set of all states over V,
- A is the set of operators O,
- $R(s, o) = \mathit{reward}(o)(s)$ for all $o \in O$ and $s \in S$,
- $T(s, o, s') = \begin{cases} p & \text{if } o \text{ applicable in } s \text{ and } \langle p, s' \rangle \in s[\![o]\!] \\ 0 & \text{otherwise} \end{cases}$
- $s_0 = I$, and
- $\gamma = d$.

G1.3 Complexity

Complexity of Probabilistic Planning

Definition (Policy Existence)

Policy existence (POLICYEX) is the following decision problem:

GIVEN: SSP planning task Π
QUESTION: Is there a proper policy for Π?

Membership in EXP

Theorem

POLICYEX $\in$ EXP

Proof.

The number of states in an SSP planning task is exponential in the number of variables. The induced SSP can be solved in time polynomial in $|S| \cdot |A|$ via linear programming and hence in time exponential in the input size. □

EXP-completeness of Probabilistic Planning

Theorem
PolicyEx *is* EXP-*complete.*

Proof Sketch.
Membership for PolicyEx: see previous slide.

Hardness is shown by Littman (1997) by reducing the EXP-complete game G_4 to PolicyEx.

G1.4 Estimated Policy Evaluation

Large SSPs and MDPs

- Before: optimal policies and exact state-values for small SSPs and MDPs.
- Now: focus on large SSPs and MDPs
- Further algorithms not necessarily optimal (may generate suboptimal policies)

Interleaved Planning & Execution

- Number of reachable states of a policy usually exponential in the number of state variables
- For large SSPs and MDPs, policies cannot be provided explicitly.
- Solution: (possibly approximate) compact representation of policy required to describe solution
  $\Rightarrow$ not part of this lecture.
- Alternative solution: interleave planning and execution

Interleaved Planning & Execution for SSPs

Plan-execute-monitor cycle for SSP $\mathcal{T}$:

- plan action a for the current state s
- execute a
- observe new current state s'
- set $s := s'$
- repeat until $s \in S_\star$

Interleaved Planning & Execution for MDPs

Plan-execute-monitor cycle for MDP $\mathcal{T}$:

- plan action a for the current state s
- execute a
- observe new current state s'
- set $s := s'$
- repeat until discounted reward sufficiently small

Interleaved Planning & Execution in Practice

- avoids loss of precision that often comes with compact description of policy
- does not waste time with planning for states that are never reached during execution
- poor decisions can be avoided by spending more time with planning before execution
- in SSPs, this can even mean that computed policy is not proper and execution never reaches the goal
- in MDPs, it is not clear when the discounted reward is sufficiently small

Estimated Policy Evaluation

- The quality of a policy is described by the state-value of the initial state $V_\pi(s_0)$
- Quality of given policy π can be computed (via LP or backward induction) or approximated arbitrarily closely (via iterative policy evaluation) in small SSPs or MDPs
- Impossible if planning and execution are interleaved as policy is incomplete

$\Rightarrow$ Estimate quality of policy π by executing it $n \in \mathbb{N}$ times

Executing a Policy

Definition (Run in SSP)

Let $\mathcal{T}$ be an SSP and π be a proper policy for $\mathcal{T}$.
A sequence of transitions

$$\rho_\pi = s_0 \xrightarrow{p_1:\pi(s_0)} s_1, \dots, s_{n-1} \xrightarrow{p_n:\pi(s_{n-1})} s_n$$

is a run ρ_π of π if $s_{i+1} \sim s_i[\![\pi(s_i)]\!]$ and $s_n \in S_\star$.
The cost of run ρ_π is $cost(\rho_\pi) = \sum_{i=0}^{n-1} cost(\pi(s_i))$.

A run in an SSP can easily be generated by executing π from s_0 until a state $s \in S_\star$ is encountered.

Executing a Policy

Definition (Run in MDP)

Let $\mathcal{T}$ be an MDP and π be a policy for $\mathcal{T}$.
A sequence of transitions

$$\rho_\pi = s_0 \xrightarrow{p_1:\pi(s_0)} s_1, \dots, s_{n-1} \xrightarrow{p_n:\pi(s_{n-1})} s_n$$

is a run ρ_π of π if $s_{i+1} \sim s_i[\![\pi(s_i)]\!]$.
The reward of run ρ_π is $reward(\rho_\pi) = \sum_{i=0}^{n-1} \gamma^i \cdot reward(s_i, \pi(s_i))$.

To generate a run, a termination criterion (e.g., based on the change of the accumulated reward) must be specified.

Estimated Policy Evaluation

Definition (Estimated Policy Evaluation)

Let $\mathcal{T}$ be an SSP, π be a policy for $\mathcal{T}$ and $\langle \rho_\pi^1, \dots, \rho_\pi^n \rangle$ be a sequence of runs of π.
The estimated quality of π via estimated policy evaluation is

$$\tilde{V}_\pi := \frac{1}{n} \cdot \sum_{i=1}^{n} cost(\rho_\pi^i).$$

Convergence of Estimated Policy Evaluation in SSPs

Theorem

*Let $\mathcal{T}$ be an SSP, π be a policy for $\mathcal{T}$ and $\langle \rho_\pi^1, \dots, \rho_\pi^n \rangle$ be a sequence of runs of π.
Then $\tilde{V}_\pi \to V_\pi(s_0)$ for $n \to \infty$.*

Proof.

Holds due to the strong law of large numbers. □

$\Rightarrow$ $\tilde{V}_\pi$ is a good approximation of $v_\pi(s_0)$ if n sufficiently large.

G1.5 Summary

Summary

- MDP and SSP planning tasks represent MDPs and SSPs compactly.
- Policy existence in SSPs is EXP-complete.
- Interleaving planning and execution avoids representation issues of (typically exponentially sized) policy.
- Quality of such an incomplete policy can be estimated by executing it a fixed number of times.
- In SSPs, estimated policy evaluation converges to the true quality of the policy.