Planning and Optimization

E4. Linear & Integer Programming

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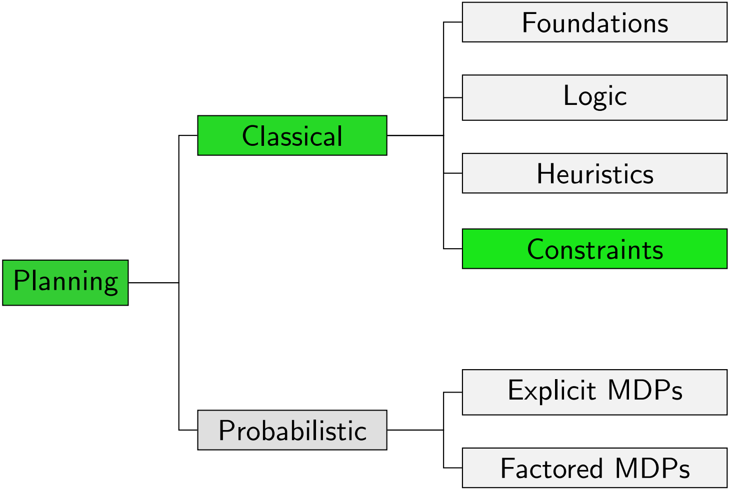
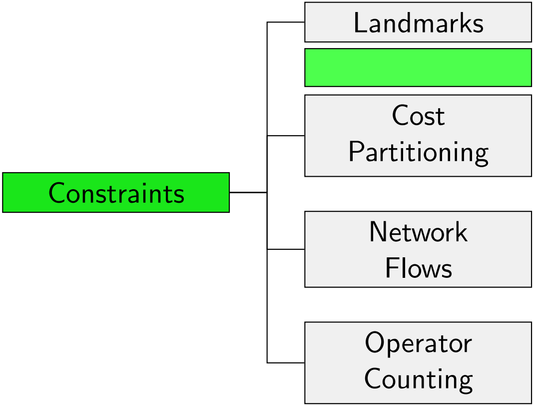
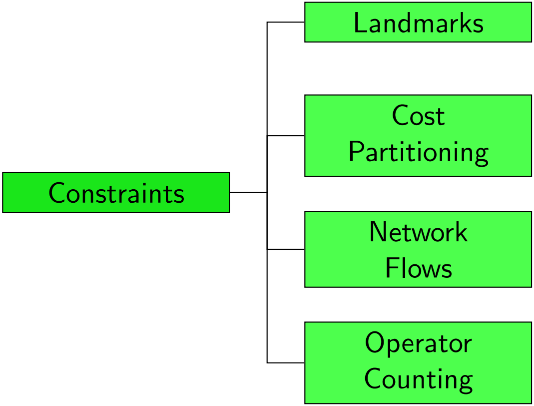
Content of this Course



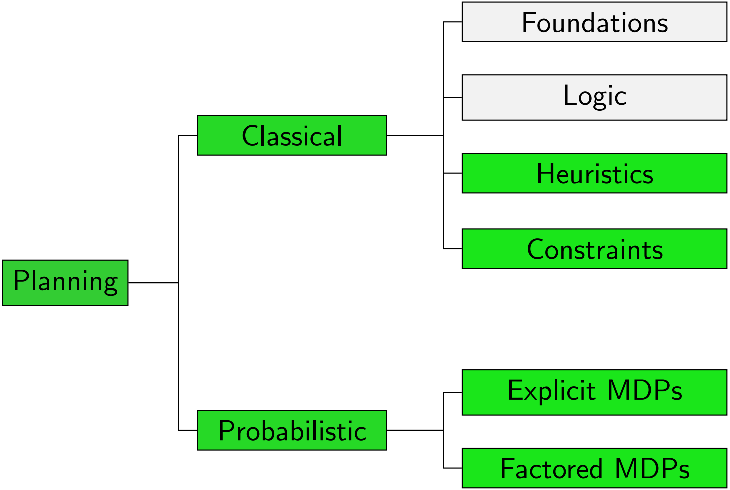
Content of this Course: Constraints (Timeline)



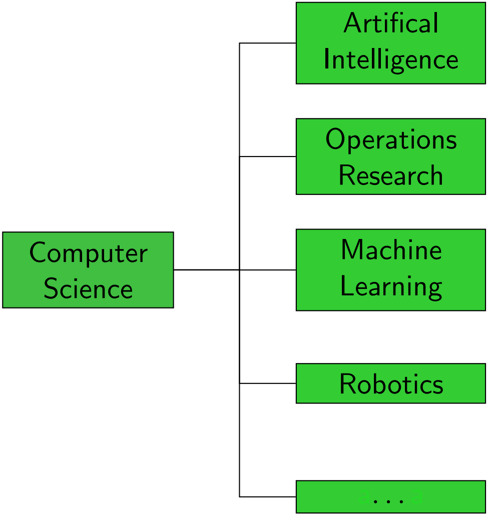
Content of this Course: Constraints (Relevance)



Content of this Course (Relevance)



Content of this Course (Relevance)



Integer Programs

Motivation

- This goes on beyond Computer Science
- Active research on IPs and LPs in
  - Operation Research
  - Mathematics
- Many application areas, for instance:
  - Manufacturing
  - Agriculture
  - Mining
  - Logistics
  - Planning
- As an application, we treat LPs / IPs as a blackbox
- We just look at the fundamentals

Motivation

Example (Optimization Problem)

Consider the following scenario:

- A factory produces two products A and B
- Selling a unit of B yields 5 times the profit of a unit of A.
- A client places the unusual order to "buy anything that can be produced on that day as long as the units of B do not exceed two plus twice the units of A."
- The factory can produce at most 12 products per day.
- There is only material for 6 units of A (there is enough material to produce any amount of B)

How many units of A and B does the client receive if the factory owner aims to maximize her profit?

Integer Program: Example

Let X_A and X_B be the (integer) number of produced A and B

Example (Optimization Problem as Integer Program)

$$X_A \geq 0, \quad X_B \geq 0$$

Example (Optimization Problem)

Integer Program: Example

Let X_A and X_B be the (integer) number of produced A and B

Example (Optimization Problem as Integer Program)

$$\text{maximize} \quad X_A + 5X_B \quad \text{subject to}$$

$$X_A \geq 0, \quad X_B \geq 0$$

Example (Optimization Problem)

- "a unit of B yields 5 times the profit of a unit of A"
- "the factory owner aims to maximize her profit"

Integer Program: Example

Let X_A and X_B be the (integer) number of produced A and B

Example (Optimization Problem as Integer Program)

maximize $\quad X_A + 5X_B \quad$ subject to

$$2 + 2X_A \geq X_B$$

$$X_A \geq 0, \quad X_B \geq 0$$

Example (Optimization Problem)

- "the units of B may not exceed two plus twice the units of A."

Integer Program: Example

Let X_A and X_B be the (integer) number of produced A and B

Example (Optimization Problem as Integer Program)

maximize $X_A + 5X_B$ subject to

$$2 + 2X_A \geq X_B$$
$$X_A + X_B \leq 12$$

$$X_A \geq 0, \quad X_B \geq 0$$

Example (Optimization Problem)

- "The factory can produce at most 12 units per day"

Integer Program: Example

Let X_A and X_B be the (integer) number of produced A and B

Example (Optimization Problem as Integer Program)

maximize $X_A + 5X_B$ subject to

$$2 + 2X_A \geq X_B$$
$$X_A + X_B \leq 12$$
$$X_A \leq 6$$

$$X_A \geq 0, \quad X_B \geq 0$$

Example (Optimization Problem)

- "There is only material for 6 units of A"

Integer Program: Example

Let X_A and X_B be the (integer) number of produced A and B

Example (Optimization Problem as Integer Program)

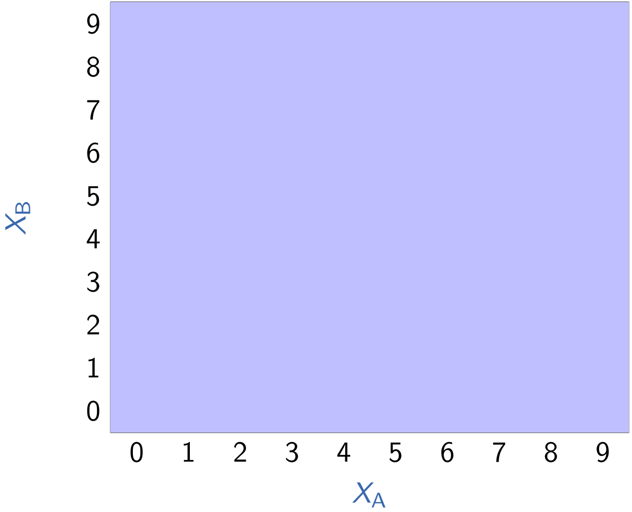
$$\text{maximize} \quad X_A + 5X_B \quad \text{subject to}$$

$$\begin{aligned} 2 + 2X_A &\geq X_B \\ X_A + X_B &\leq 12 \\ X_A &\leq 6 \end{aligned}$$

$$X_A \geq 0, \quad X_B \geq 0$$

$\rightsquigarrow$ unique optimal solution:
produce 4 A ($X_A = 4$) and 8 B ($X_B = 8$) for a profit of 44

Integer Program Example: Visualization



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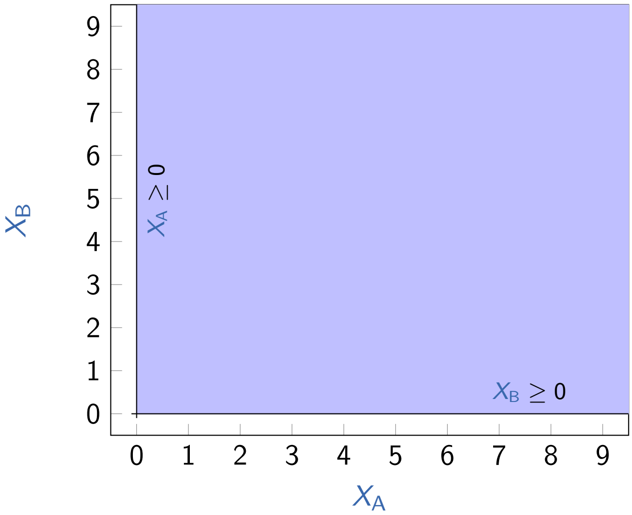

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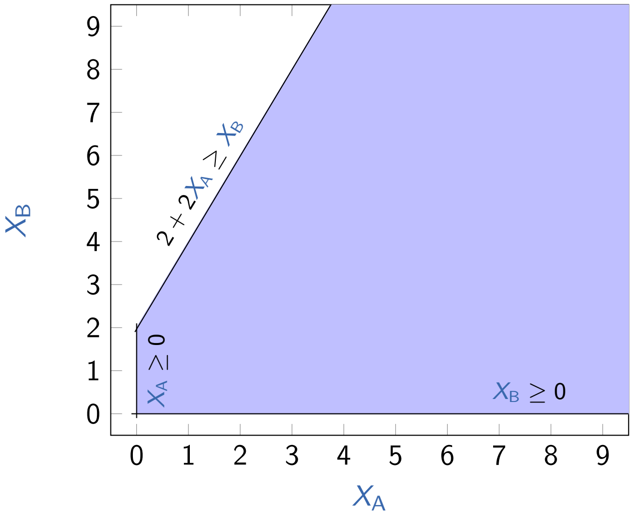

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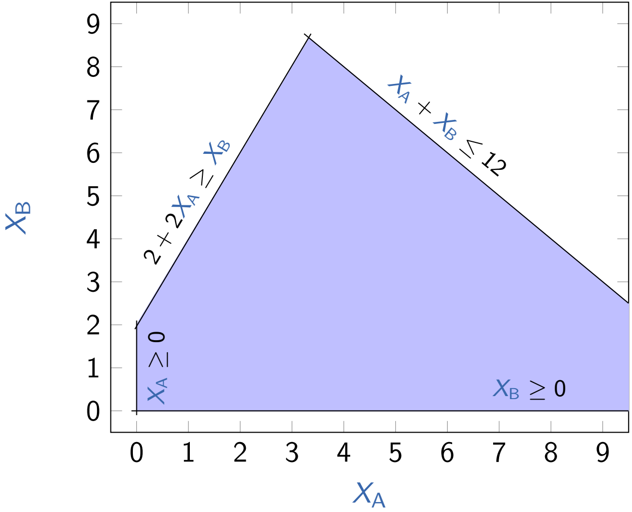

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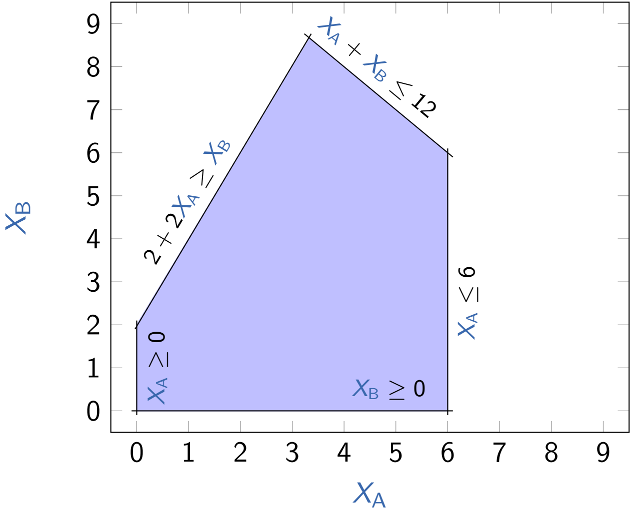

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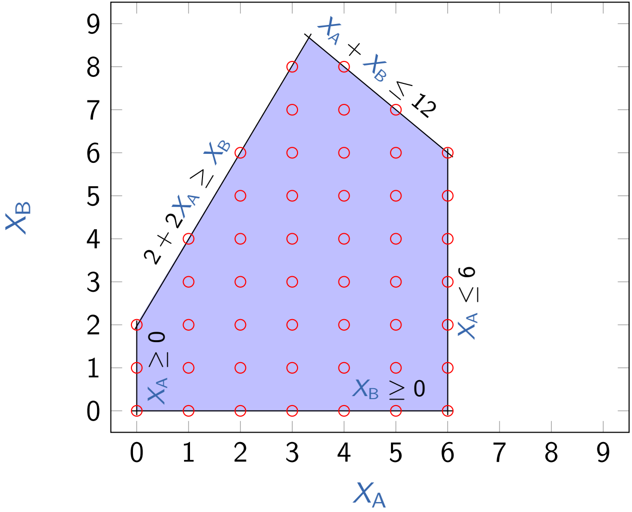
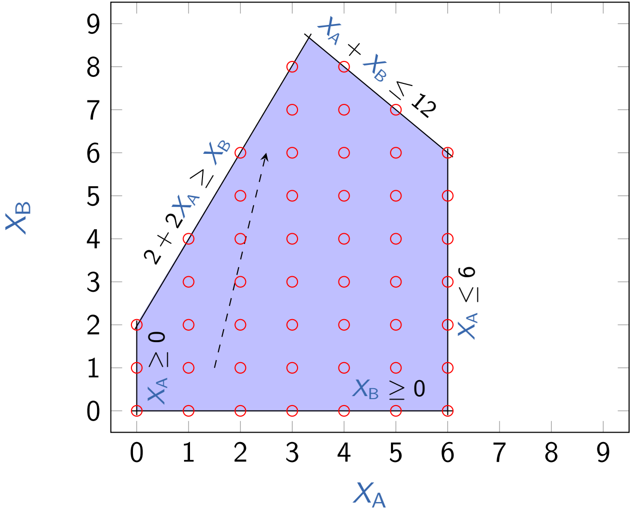
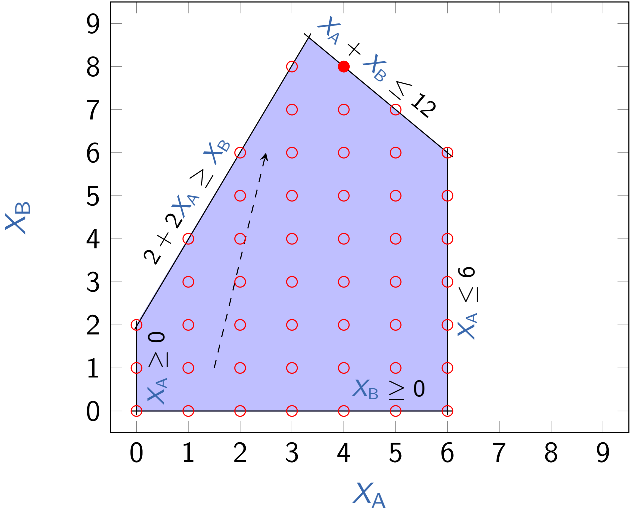


Integer Program Example: Visualization



Integer Program Example: Visualization



Integer Programs

Integer Program

An integer program (IP) consists of:

- a finite set of integer-valued variables V
- a finite set of linear inequalities (constraints) over V
- an objective function, which is a linear combination of V
- which should be minimized or maximized.

Terminology

- An integer assignment to all variables in V is feasible if it satisfies the constraints.
- An integer program is feasible if there is such a feasible assignment. Otherwise it is infeasible.
- A feasible maximum (resp. minimum) problem is unbounded if the objective function can assume arbitrarily large positive (resp. negative) values at feasible assignments. Otherwise it is bounded.
- The objective value of a bounded feasible maximum (resp. minimum) problem is the maximum (resp. minimum) value of the objective function with a feasible assignment.

Three classes of IPs

IPs fall into three classes:

- bounded feasible: IP is solvable and optimal solutions exist
- unbounded feasible: IP is solvable and arbitrarily good solutions exist
- infeasible: IP is unsolvable

Another Example

Example

minimize $\quad 3X_{o_1} + 4X_{o_2} + 5X_{o_3} \quad$ subject to

$$\begin{aligned} X_{o_4} &\geq 1 \\ X_{o_1} + X_{o_2} &\geq 1 \\ X_{o_1} + X_{o_3} &\geq 1 \\ X_{o_2} + X_{o_3} &\geq 1 \end{aligned}$$

$$X_{o_1} \geq 0, \quad X_{o_2} \geq 0, \quad X_{o_3} \geq 0, \quad X_{o_4} \geq 0$$

What example from a previous chapter does this IP encode?

Another Example

Example

minimize $3X_{o_1} + 4X_{o_2} + 5X_{o_3}$ subject to

$$X_{o_4} \geq 1$$
$$X_{o_1} + X_{o_2} \geq 1$$
$$X_{o_1} + X_{o_3} \geq 1$$
$$X_{o_2} + X_{o_3} \geq 1$$

$$X_{o_1} \geq 0, \quad X_{o_2} \geq 0, \quad X_{o_3} \geq 0, \quad X_{o_4} \geq 0$$

What example from a previous chapter does this IP encode?

$\rightsquigarrow$ the minimum hitting set from Chapter E2

Complexity of solving Integer Programs

- As an IP can compute an MHS, solving an IP must be at least as complex as computing an MHS
- Reminder: MHS is a “classical” NP-complete problem
- Good news: Solving an IP is not harder

$\rightsquigarrow$ Finding solutions for IPs is NP-complete.

Complexity of solving Integer Programs

- As an IP can compute an MHS, solving an IP must be at least as complex as computing an MHS
- Reminder: MHS is a "classical" NP-complete problem
- Good news: Solving an IP is not harder

⇝ Finding solutions for IPs is NP-complete.

Removing the requirement that solutions must be integer-valued leads to a simpler problem

Linear Programs

Linear Programs

Linear Program

A linear program (LP) consists of:

- a finite set of real-valued variables V
- a finite set of linear inequalities (constraints) over V
- an objective function, which is a linear combination of V
- which should be minimized or maximized.

We use the introduced IP terminology also for LPs.

Mixed IPs (MIPs) generalize IPs and LPs:
some variables are integer-values, some are real-valued.

Linear Program: Example

Let X_A and X_B be the (real-valued) number of produced A and B

Example (Optimization Problem as Linear Program)

maximize $X_A + 5X_B$ subject to

$$2 + 2X_A \geq X_B$$
$$X_A + X_B \leq 12$$
$$X_A \leq 6$$

$$X_A \geq 0, \quad X_B \geq 0$$

Linear Program: Example

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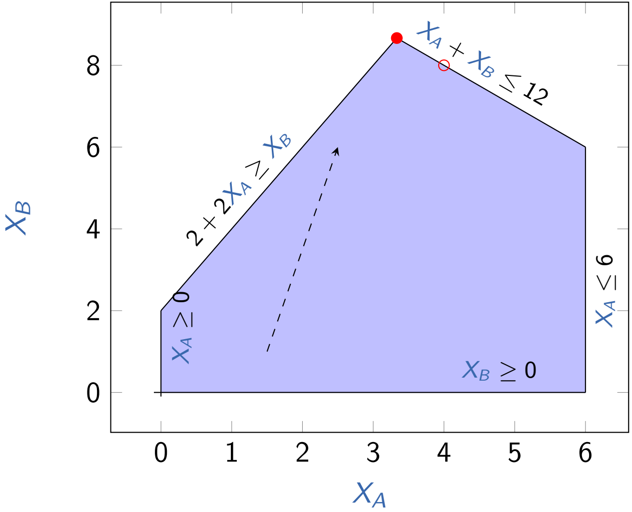
maximize $X_A + 5X_B$ subject to

$$
\begin{aligned}
2 + 2X_A &\geq X_B \\
X_A + X_B &\leq 12 \\
X_A &\leq 6
\end{aligned}
$$

$$X_A \geq 0, \quad X_B \geq 0$$

$\rightsquigarrow$ unique optimal solution:
$X_A = 3\frac{1}{3}$ and $X_B = 8\frac{2}{3}$ with objective value $46\frac{2}{3}$

Linear Program Example: Visualization



Solving Linear Programs

- Observation:
  For an maximization problem, the objective value of the LP is not lower than the one of the IP.

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  LP solving is a polynomial-time problem.

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- Observation:
  For an maximization problem, the objective value of the LP is not lower than the one of the IP.
- Complexity:
  LP solving is a polynomial-time problem.
- Common idea:
  Approximate IP objective value with corresponding LP (LP relaxation).

LP Relaxation

Theorem (LP Relaxation)

The LP relaxation of an integer program is the problem that arises by removing the requirement that variables are integer-valued.

For a maximization (resp. minimization) problem, the objective value of the LP relaxation is an upper (resp. lower) bound on the value of the IP.

Proof idea.

Every feasible assignment for the IP is also feasible for the LP. □

LP Relaxation of MHS heuristic

Example (Minimum Hitting Set)

minimize $3X_{o_1} + 4X_{o_2} + 5X_{o_3}$ subject to

$$X_{o_4} \geq 1$$
$$X_{o_1} + X_{o_2} \geq 1$$
$$X_{o_1} + X_{o_3} \geq 1$$
$$X_{o_2} + X_{o_3} \geq 1$$

$$X_{o_1} \geq 0, \quad X_{o_2} \geq 0, \quad X_{o_3} \geq 0, \quad X_{o_4} \geq 0$$

- ⇝ optimal solution of LP relaxation:
  $X_{o_4} = 1$ and $X_{o_1} = X_{o_2} = X_{o_3} = 0.5$ with objective value 6
- ⇝ LP relaxation of MHS heuristic is admissible
  and can be computed in polynomial time

Some LP Theory: Duality

Every LP has an alternative view (its dual LP).

- roughly: variables and constraints swap roles
- roughly: objective coefficients and bounds swap roles
- dual of maximization LP is minimization LP and vice versa
- dual of dual: original LP

Duality Theorem

Theorem (Duality Theorem)

If a linear program is bounded feasible, then so is its dual, and their objective values are equal.

(Proof omitted.)

The dual provides a different perspective on a problem.

Summary

Summary

- Linear (and integer) programs consist of an objective function that should be maximized or minimized subject to a set of given linear constraints.
- Finding solutions for integer programs is NP-complete.
- LP solving is a polynomial time problem.
- The dual of a maximization LP is a minimization LP and vice versa.
- The dual of a bounded feasible LP has the same objective value.

Further Reading

The slides in this chapter are based on the following excellent tutorial on LP solving:

Thomas S. Ferguson.
Linear Programming – A Concise Introduction.
UCLA, unpublished document available online.