

Planning and Optimization

E4. Linear & Integer Programming

Malte Helmert and Gabriele Röger

Universität Basel

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Planning and Optimization

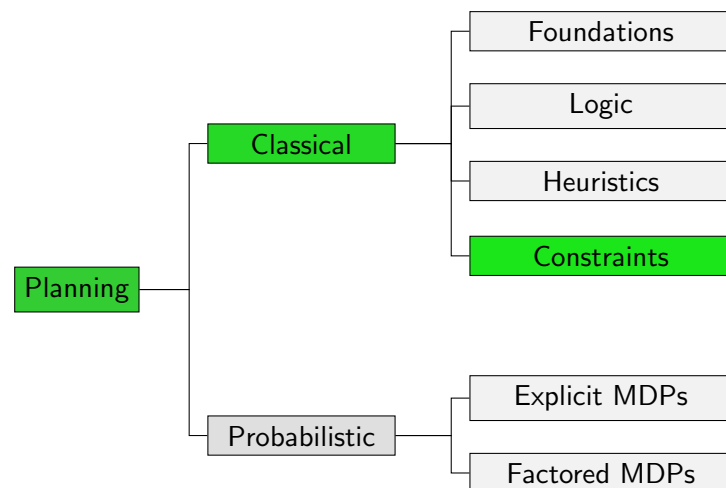
November 18, 2020 — E4. Linear & Integer Programming

E4.1 Integer Programs

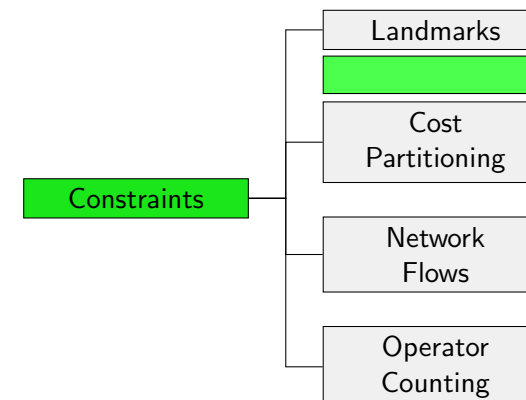
E4.2 Linear Programs

E4.3 Summary

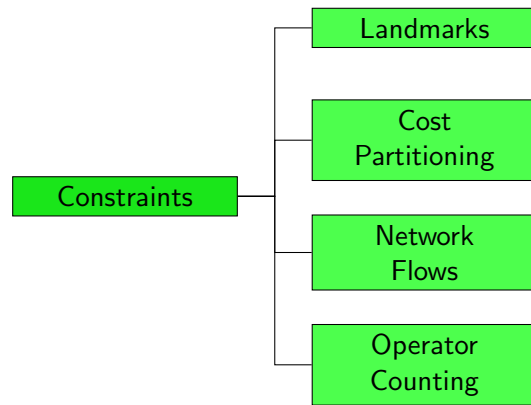
Content of this Course



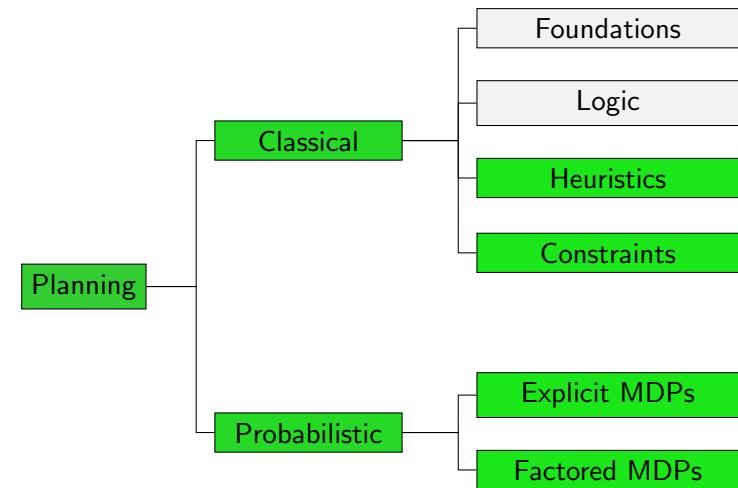
Content of this Course: Constraints (Timeline)



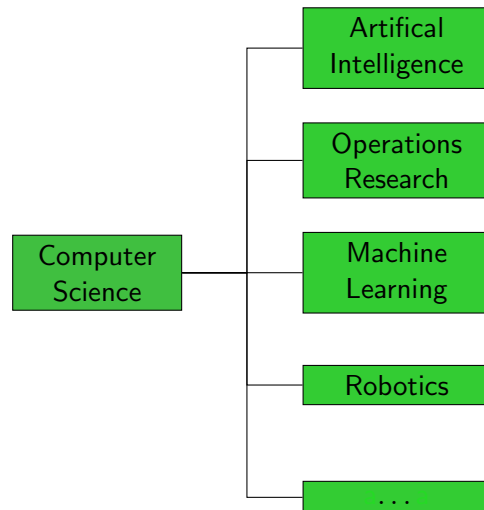
Content of this Course: Constraints (Relevance)



Content of this Course (Relevance)



Content of this Course (Relevance)



E4.1 Integer Programs

Motivation

- ▶ This goes on beyond Computer Science
- ▶ Active **research** on IPs and LPs in
 - ▶ Operation Research
 - ▶ Mathematics
- ▶ Many **application** areas, for instance:
 - ▶ Manufacturing
 - ▶ Agriculture
 - ▶ Mining
 - ▶ Logistics
 - ▶ **Planning**
- ▶ As an application, we treat LPs / IPs as a **blackbox**
- ▶ We just look at **the fundamentals**

Motivation

Example (Optimization Problem)

Consider the following scenario:

- ▶ A factory produces two products A and B
- ▶ Selling a unit of B yields 5 times the profit of a unit of A.
- ▶ A client places the unusual order to “buy anything that can be produced on that day as long as the units of B do not exceed two plus twice the units of A.”
- ▶ The factory can produce at most 12 products per day.
- ▶ There is only material for 6 units of A (there is enough material to produce any amount of B)

How many units of A and B does the client receive if the factory owner aims to maximize her profit?

Integer Program: Example

Let X_A and X_B be the (**integer**) number of produced A and B

Example (Optimization Problem as Integer Program)

maximize $X_A + 5X_B$ subject to

$$2 + 2X_A \geq X_B$$

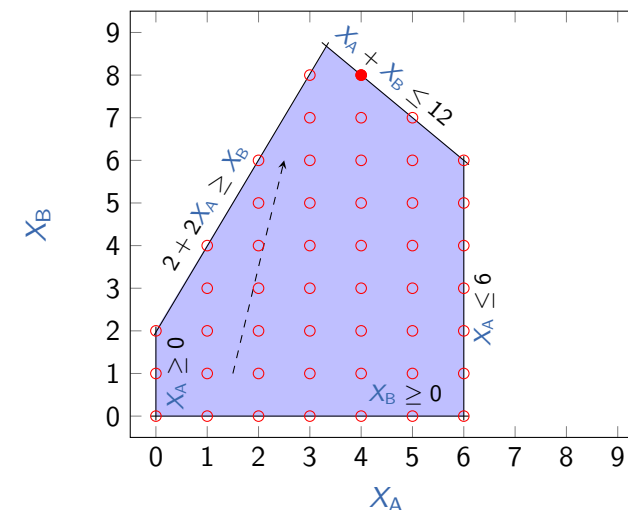
$$X_A + X_B \leq 12$$

$$X_A \leq 6$$

$$X_A \geq 0, \quad X_B \geq 0$$

↪ **unique optimal solution:**
produce 4 A ($X_A = 4$) and 8 B ($X_B = 8$) for a profit of 44

Integer Program Example: Visualization



Integer Programs

Integer Program

An **integer program (IP)** consists of:

- ▶ a finite set of **integer-valued variables** V
- ▶ a finite set of **linear inequalities** (constraints) over V
- ▶ an **objective function**, which is a linear combination of V
- ▶ which should be **minimized** or **maximized**.

Terminology

- ▶ An integer assignment to all variables in V is **feasible** if it satisfies the constraints.
- ▶ An integer program is **feasible** if there is such a feasible assignment. Otherwise it is **infeasible**.
- ▶ A feasible maximum (resp. minimum) problem is **unbounded** if the objective function can assume arbitrarily large positive (resp. negative) values at feasible assignments. Otherwise it is **bounded**.
- ▶ The **objective value** of a bounded feasible maximum (resp. minimum) problem is the maximum (resp. minimum) value of the objective function with a feasible assignment.

Three classes of IPs

IPs fall into three classes:

- ▶ **bounded feasible**: IP is solvable and optimal solutions exist
- ▶ **unbounded feasible**: IP is solvable and arbitrarily good solutions exist
- ▶ **infeasible**: IP is unsolvable

Another Example

Example

minimize $3X_{01} + 4X_{02} + 5X_{03}$ subject to

$$X_{04} \geq 1$$

$$X_{01} + X_{02} \geq 1$$

$$X_{01} + X_{03} \geq 1$$

$$X_{02} + X_{03} \geq 1$$

$$X_{01} \geq 0, \quad X_{02} \geq 0, \quad X_{03} \geq 0, \quad X_{04} \geq 0$$

What example from a previous chapter does this IP encode?

↪ the **minimum hitting set** from Chapter E2

Complexity of solving Integer Programs

- ▶ As an IP can compute an MHS, solving an IP must be **at least as complex** as computing an MHS
- ▶ Reminder: MHS is a “classical” NP-complete problem
- ▶ Good news: Solving an IP is **not harder**

↪ Finding solutions for IPs is **NP-complete**.

Removing the requirement that solutions must be **integer-valued** leads to a simpler problem

E4.2 Linear Programs

Linear Programs

Linear Program

A **linear program (LP)** consists of:

- ▶ a finite set of **real-valued variables** V
- ▶ a finite set of **linear inequalities** (constraints) over V
- ▶ an **objective function**, which is a linear combination of V
- ▶ which should be **minimized** or **maximized**.

We use the introduced IP terminology also for LPs.

Mixed IPs (MIPs) generalize IPs and LPs:
some variables are integer-values, some are real-valued.

Linear Program: Example

Let X_A and X_B be the (**real-valued**) number of produced A and B

Example (Optimization Problem as Linear Program)

maximize $X_A + 5X_B$ subject to

$$2 + 2X_A \geq X_B$$

$$X_A + X_B \leq 12$$

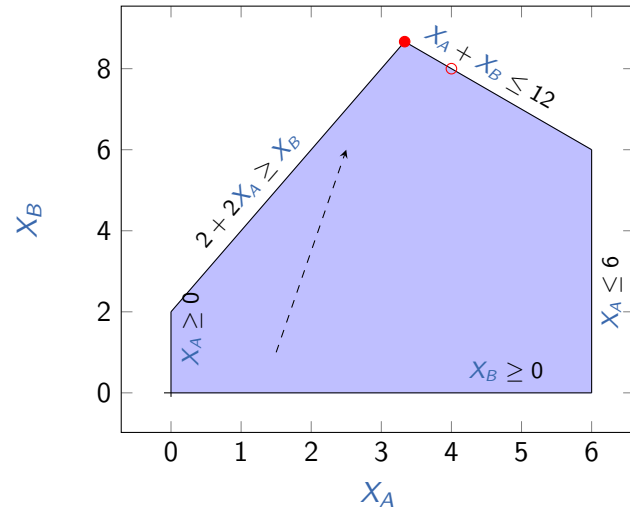
$$X_A \leq 6$$

$$X_A \geq 0, \quad X_B \geq 0$$

↪ unique optimal solution:

$$X_A = 3\frac{1}{3} \text{ and } X_B = 8\frac{2}{3} \text{ with objective value } 46\frac{2}{3}$$

Linear Program Example: Visualization



Solving Linear Programs

- ▶ **Observation:**
For an maximization problem, the objective value of the LP is **not lower** than the one of the IP.
- ▶ **Complexity:**
LP solving is a **polynomial-time** problem.
- ▶ **Common idea:**
Approximate IP objective value with corresponding LP (**LP relaxation**).

LP Relaxation

Theorem (LP Relaxation)

The **LP relaxation** of an integer program is the problem that arises by removing the requirement that variables are integer-valued.

For a **maximization** (resp. **minimization**) problem, the objective value of the LP relaxation is an **upper** (resp. **lower**) **bound** on the value of the IP.

Proof idea.

Every feasible assignment for the IP is also feasible for the LP. \square

LP Relaxation of MHS heuristic

Example (Minimum Hitting Set)

minimize $3X_{o_1} + 4X_{o_2} + 5X_{o_3}$ subject to

$$X_{o_4} \geq 1$$

$$X_{o_1} + X_{o_2} \geq 1$$

$$X_{o_1} + X_{o_3} \geq 1$$

$$X_{o_2} + X_{o_3} \geq 1$$

$$X_{o_1} \geq 0, \quad X_{o_2} \geq 0, \quad X_{o_3} \geq 0, \quad X_{o_4} \geq 0$$

\rightsquigarrow optimal solution of LP relaxation:

$X_{o_4} = 1$ and $X_{o_1} = X_{o_2} = X_{o_3} = 0.5$ with objective value 6

\rightsquigarrow LP relaxation of MHS heuristic is **admissible** and can be computed **in polynomial time**

Some LP Theory: Duality

Every LP has an alternative view (its **dual** LP).

- ▶ roughly: variables and constraints swap roles
- ▶ roughly: objective coefficients and bounds swap roles
- ▶ dual of maximization LP is minimization LP and vice versa
- ▶ dual of dual: original LP

Duality Theorem

Theorem (Duality Theorem)

*If a linear program is **bounded feasible**, then so is its dual, and their **objective values are equal**.*

(Proof omitted.)

The dual provides a different perspective on a problem.

E4.3 Summary

Summary

- ▶ **Linear (and integer) programs** consist of an **objective function** that should be **maximized or minimized** subject to a set of given **linear constraints**.
- ▶ Finding solutions for **integer** programs is **NP-complete**.
- ▶ **LP solving** is a **polynomial time** problem.
- ▶ The dual of a maximization LP is a minimization LP and vice versa.
- ▶ The **dual** of a bounded feasible LP has the **same objective value**.

Further Reading

The slides in this chapter are based on the following excellent tutorial on LP solving:



Thomas S. Ferguson.

Linear Programming – A Concise Introduction.

[UCLA, unpublished document available online.](#)