

# Discrete Mathematics in Computer Science

## C4. Further Topics in Graph Theory

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## C4.1 Subgraphs

## C4.2 Isomorphism

## C4.3 Planarity and Minors

## C4.1 Subgraphs

## Overview

- ▶ We conclude our discussion of (di-) graphs by giving a brief tour of some further topics in graph theory that we do not have time to discuss in depth.
- ▶ In the interest of brevity (and hence wider coverage of topics), we do not give proofs for the results in this chapter.

## Subgraphs

### Definition (subgraph)

A **subgraph** of a graph  $(V, E)$  is a graph  $(V', E')$  with  $V' \subseteq V$  and  $E' \subseteq E$ .

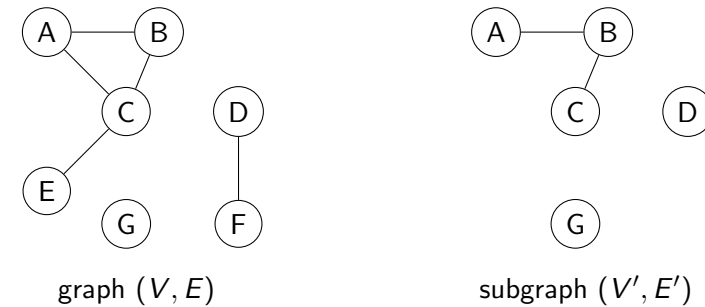
A **subgraph** of a digraph  $(N, A)$  is a digraph  $(N', A')$  with  $N' \subseteq N$  and  $A' \subseteq A$ .

**German:** Teilgraph/Untergraph

**Question:** Can we choose  $V'$  and  $E'$  arbitrarily?

The subgraph relationship defines a **partial order** on graphs (and on digraphs).

## Subgraphs – Example



## Induced Subgraphs (1)

### Definition (induced subgraph)

Let  $G = (V, E)$  be a graph, and let  $V' \subseteq V$ .

The **subgraph of  $G$  induced by  $V'$**  is the graph  $(V', E')$  with  $E' = \{\{u, v\} \in E \mid u, v \in V'\}$ .

We say that  $G'$  is an **induced subgraph** of  $G = (V, E)$  if  $G'$  is the subgraph of  $G$  induced by  $V'$  for any set of vertices  $V' \subseteq V$ .

**German:** induzierter Teilgraph (eines Graphen)

## Induced Subgraphs (2)

### Definition (induced subgraph)

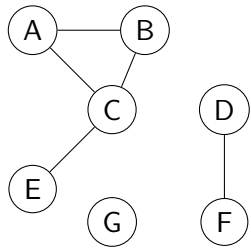
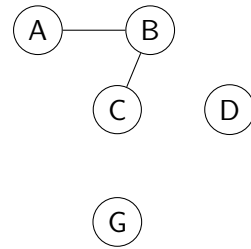
Let  $G = (N, A)$  be a digraph, and let  $N' \subseteq N$ .

The **subgraph of  $G$  induced by  $N'$**  is the digraph  $(N', A')$  with  $A' = \{(u, v) \in A \mid u, v \in N'\}$ .

We say that  $G'$  is an **induced subgraph** of  $G = (N, A)$  if  $G'$  is the subgraph of  $G$  induced by  $N'$  for any set of nodes  $N' \subseteq N$ .

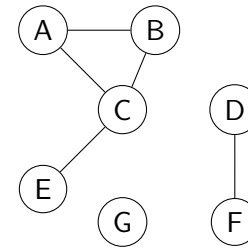
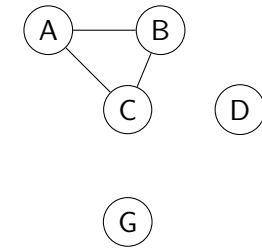
**German:** induzierter Teilgraph (eines gerichteten Graphen)

## Induced Subgraphs – Example

graph  $(V, E)$ 

Is this an induced subgraph?

## Induced Subgraphs – Example

graph  $(V, E)$ 

This is an induced subgraph.

## Induced Subgraphs – Discussion

- ▶ Induced subgraphs are subgraphs.
- ▶ They are the **largest** (in terms of the set of edges) subgraphs with any given set of vertices.
- ▶ A typical example are subgraphs induced by the connected components of a graph.
- ▶ The subgraphs induced by the connected components of a forest are trees.

## Counting Subgraphs

- ▶ How many subgraphs does a graph  $(V, E)$  have?
- ▶ How many induced subgraph does a graph  $(V, E)$  have?

For the second question, the answer is  $2^{|V|}$ .

The first question is in general not easy to answer because vertices and edges of a subgraph cannot be chosen independently.

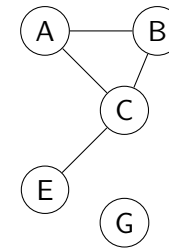
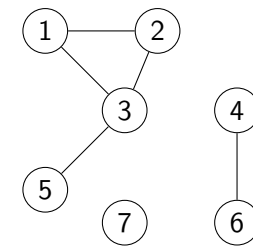
### Example (subgraphs of a complete graph)

A **complete** graph with  $n$  vertices (i.e., with all possible  $\binom{n}{2}$  edges) has  $\sum_{k=0}^n \binom{n}{k} 2^{\binom{k}{2}}$  subgraphs. (Why?)

for  $n = 10$ : 1024 induced subgraphs, 35883905263781 subgraphs

## C4.2 Isomorphism

## Motivation

graph  $(V, E)$ graph  $(V', E')$ 

What is the difference between these graphs?

## Isomorphism

- ▶ In many cases, the “names” of the vertices of a graph do not have any particular semantic meaning.
- ▶ Often, we care about the **structure** of the graph, i.e., the relationship between the vertices and edges, but not what we **call** the different vertices.
- ▶ This is captured by the concept of **isomorphism**.

## Isomorphism – Definition

### Definition (Isomorphism)

Let  $G = (V, E)$  and  $G' = (V', E')$  be graphs.

An **isomorphism** from  $G$  to  $G'$  is a **bijective** function  $\sigma : V \rightarrow V'$  such that for all  $u, v \in V$ :

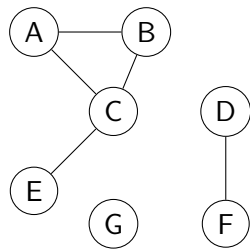
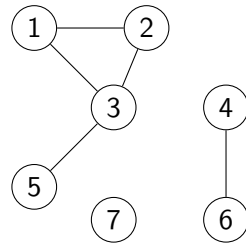
$$\{u, v\} \in E \quad \text{iff} \quad \{\sigma(u), \sigma(v)\} \in E'.$$

If there exists an isomorphism from  $G$  to  $G'$ , we say that they are **isomorphic**, in symbols  $G \cong G'$ .

**German:** Isomorphismus, isomorph

- ▶ derives from Ancient Greek for “equally shaped/formed”
- ▶ analogous definition for digraphs omitted

## Isomorphism – Example

graph  $(V, E)$ graph  $(V', E')$ 

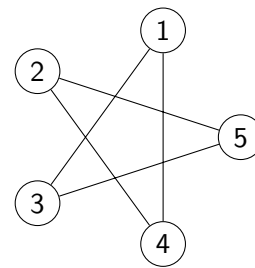
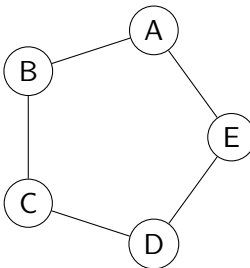
- ▶  $\sigma = \{A \mapsto 1, B \mapsto 2, C \mapsto 3, D \mapsto 4, E \mapsto 5, F \mapsto 6, G \mapsto 7\}$
- ▶ for example:  $\{A, B\} \in E$  and  $\{\sigma(A), \sigma(B)\} = \{1, 2\} \in E'$
- ▶ for example:  $\{A, D\} \notin E$  and  $\{\sigma(A), \sigma(D)\} = \{1, 4\} \notin E'$

## Isomorphism – Discussion

- ▶ The **identity function** is an isomorphism.
- ▶ The **inverse** of an isomorphism is an isomorphism.
- ▶ The **composition** of two isomorphisms is an isomorphism (when defined over matching sets of vertices)

It follows that being isomorphic is an **equivalence relation**.

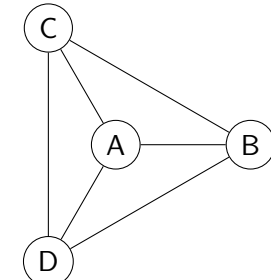
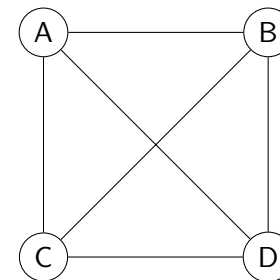
## Isomorphic or Not? (1)



**isomorphic**

$$\sigma = \{A \mapsto 1, B \mapsto 3, C \mapsto 5, D \mapsto 2, E \mapsto 4\}$$

## Isomorphic or Not? (2)

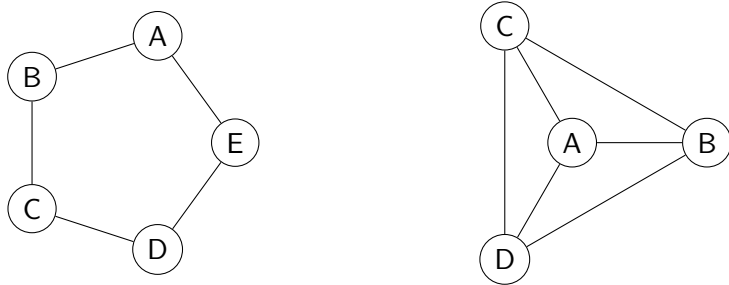


**isomorphic**

$\rightsquigarrow$  in fact, **the same graph!**

$$\sigma = \{A \mapsto A, B \mapsto B, C \mapsto C, D \mapsto D\}$$

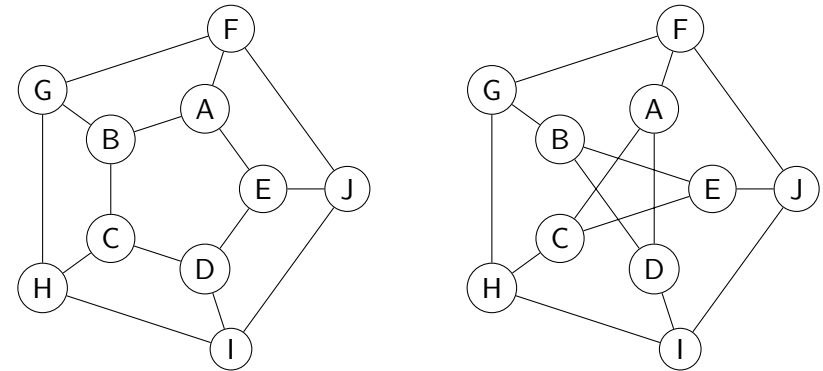
## Isomorphic or Not? (3)



not isomorphic

There does not even exist a bijection between the vertices.

## Isomorphic or Not? (4)

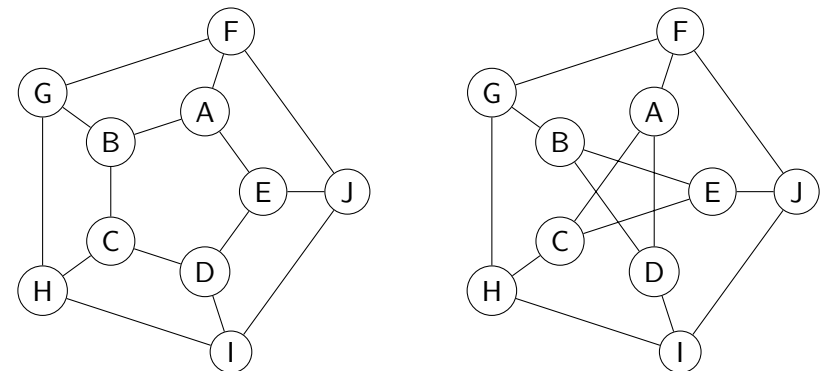


isomorphic or not?

## Proving and Disproving Isomorphism

- ▶ To prove that two graphs **are** isomorphic, it suffices to state an isomorphism and verify that it has the required properties.
- ▶ To prove that two graphs are **not** isomorphic, we must rule out all possible bijections.
  - ▶ With  $n$  vertices, there are  $n!$  bijections.
  - ▶ example  $n = 10$ :  $10! = 3628800$
- ▶ A common disproof idea is to identify a **graph invariant**, i.e., a property of a graph that must be **the same** in isomorphic graphs, and show that it differs.
  - ▶ examples: number of vertices, number of edges, maximum/minimum degree, sorted sequence of all degrees, number of connected components

## Isomorphic or Not? (5)



not isomorphic

- ▶ The left graph has cycles of length 4 (e.g.,  $\langle A, B, G, F, A \rangle$ ).
- ▶ The right graph does not.
- ▶ Having a cycle of a given length is an invariant.

## Scientific Pop Culture

- ▶ Determining if two graphs are isomorphic is an algorithmic problem that has been famously resistant to studying its complexity.
- ▶ For more than 40 years, we have not known if polynomial algorithms exist, and we also do not know if it belongs to the famous class of **NP-complete** problems.
- ▶ In 2015, László Babai announced an algorithm with **quasi-polynomial** (worse than polynomial, better than exponential) runtime.

### Further Reading

Martin Grohe, Pascal Schweitzer.

[The Graph Isomorphism Problem.](#)

Communications of the ACM 63(11):128–134, November 2020.

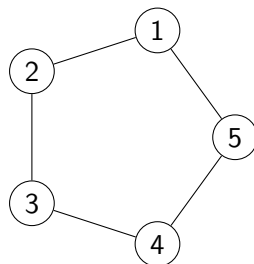
<https://dl.acm.org/doi/10.1145/3372123>

## Symmetries, Automorphisms and Group Theory

- ▶ An isomorphism  $\sigma$  between a graph  $G$  and itself is called an **automorphism** or **symmetry** of  $G$ .
- ▶ For every graph, its symmetries are permutations of its vertex set that form a **group** (with function composition as the binary operation) called the **automorphism group** of the graph.

**Example:** the symmetric group  $S_n$  is the automorphism group of the complete graph with the vertices  $\{1, \dots, n\}$ .

## Automorphism Group of a Graph



What are the symmetries?

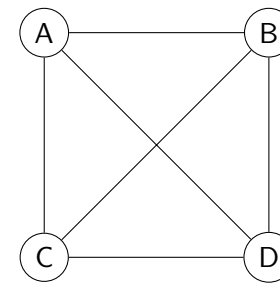
- ▶ one example is the **rotation**  
 $\sigma_1 = \{1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 4, 4 \mapsto 5, 5 \mapsto 1\}$
- ▶ another example is the **reflection**  
 $\sigma_2 = \{1 \mapsto 5, 2 \mapsto 4, 3 \mapsto 3, 4 \mapsto 2, 5 \mapsto 1\}$
- ▶ There are 10 symmetries in total, and they are all **generated** by  $\sigma_1$  and  $\sigma_2$ .

## C4.3 Planarity and Minors

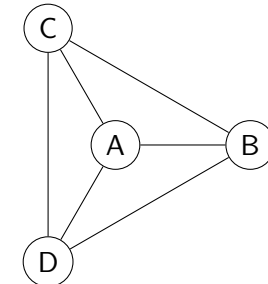
## Planarity

- ▶ We often draw graphs as 2-dimensional pictures.
- ▶ When we do so, we usually try to draw them in such a way that different edges do not cross.
- ▶ This often makes the picture neater and the edges easier to visualize.
- ▶ A picture of a graph with no edge crossings is called a **planar embedding**.
- ▶ A graph for which a planar embedding exists is called **planar**.

## Planar Embeddings – Example



not a planar embedding



planar embedding

The complete graph over 4 vertices is planar.

## Planar Graphs

### Definition (planar)

A graph  $G = (V, E)$  is called **planar** if there exists a **planar embedding** of  $G$ , i.e., a picture of  $G$  in the Euclidean plane in which no two edges intersect.

German: planar

### Notes:

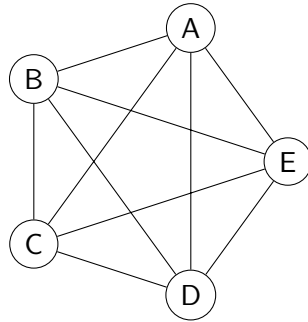
- ▶ We do not formally define planar embeddings, as this is nontrivial and not necessary for our discussion.
- ▶ In general, we may draw edges as arbitrary curves.
- ▶ However, it is possible to show that a graph has a planar embedding iff it has a planar embedding where all edges are **straight lines**.

## Planar Graphs – Discussion

- ▶ Planar graphs arise in many practical applications.
- ▶ Many computational problems are **easier** for planar graphs.
  - ▶ For example, every planar graph can be **coloured** with at most 4 colours (i.e., we can assign one of four colours to each vertex such that two neighbours always have different colours).
- ▶ For this reason, planarity is of great practical interest.
- ▶ How can we **recognize** that a graph is planar?
- ▶ How can we prove that a graph is **not** planar?

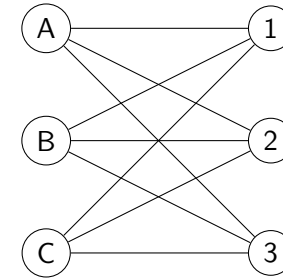


## Planar Graphs – Counterexample (1)



The complete graph  $K_5$  over 5 vertices is not planar.  
(We do not prove this result.)

## Planar Graphs – Counterexample (2)



The complete bipartite graph  $K_{3,3}$  over 3 + 3 vertices is not planar.  
(We do not prove this result.)

## Non-Planarity in General

- ▶ The two non-planar graphs  $K_5$  and  $K_{3,3}$  are special: they are the **smallest** non-planar graphs.
- ▶ In fact, something much more powerful holds: a graph is planar **iff** it does not **contain**  $K_5$  or  $K_{3,3}$ .
- ▶ The notion of **containment** we need here is related to the notion of subgraphs that we introduced, but a bit more complex. We will discuss it next.

## Edge Contraction

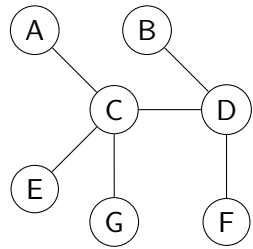
We say that  $G' = (V', E')$  can be obtained from graph  $G = (V, E)$  by **contracting the edge**  $\{u, v\} \in E$  if

- ▶  $V' = (V \setminus \{u, v\}) \cup \{uv\}$ , where  $uv \notin V$  is a new vertex
- ▶  $E' = \{e \in E \mid e \cap \{u, v\} = \emptyset\} \cup \{\{uv, w\} \mid \{u, w\} \in E \text{ or } \{v, w\} \in E\}$ .

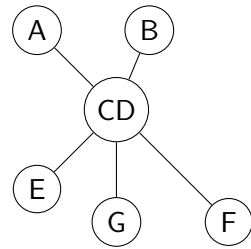
In words, we **combine** the vertices  $u$  and  $v$  (which must be connected by an edge) into a single vertex  $uv$ .

The neighbours of  $uv$  are the union of the neighbours of  $u$  and the neighbours of  $v$ .

## Edge Contraction – Example



graph  $(V, E)$



after contracting  $\{C, D\}$

## Minor

### Definition (minor)

We say that a graph  $G'$  is a **minor** of a graph  $G$  if it can be obtained from  $G$  through a sequence of transformations of the following kind:

- 1 remove a vertex (of degree 0) from the graph
- 2 remove an edge from the graph
- 3 contract an edge in the graph

German: Minor (plural: Minoren)

### Notes:

- ▶ If we only allowed the first two transformations, we would obtain the regular subgraph relationship.
- ▶ It follows that every subgraph is a minor, but the opposite is not true in general.

## Wagner's Theorem

### Theorem (Wagner's Theorem)

*A graph is planar iff it does not contain  $K_5$  or  $K_{3,3}$  as a minor.*

German: Satz von Wagner

Note: There exist linear algorithms for testing planarity.

## Minor-Hereditary Properties

- ▶ Being planar is what is called a **minor-hereditary** property: if  $G$  is planar, then all its minors are also planar.
- ▶ There exist many other important such properties.
- ▶ One example is acyclicity.

How could one prove that a property is minor-hereditary?

## The Graph Minor Theorem

### Theorem (Graph minor theorem)

Let  $\Pi$  be a minor-hereditary properties of graphs.

Then there exists a finite set of **forbidden minors**  $F(\Pi)$  such that the following result holds:

A graph has property  $\Pi$  iff it does not have any graph from  $F(\Pi)$  as a minor.

German: Minorentheorem

Examples:

- ▶ the forbidden minors for **planarity** are  $K_5$  and  $K_{3,3}$
- ▶ the (only) forbidden minor for **acyclicity** is  $K_3$ , the complete graph with 3 vertices (a.k.a. the 3-cycle graph)

## Remarks on the Graph Minor Theorem (1)

- ▶ The graph minor theorem is also known as the **Robertson-Seymour theorem**.
- ▶ It was proved by Robertson and Seymour in a series of 20 papers between 1983–2004, totalling 500+ pages.
- ▶ It is one of the most important results in graph theory.

## Remarks on the Graph Minor Theorem (2)

- ▶ In principle, for every **fixed** graph  $H$ , we can test if  $H$  is a minor of a graph  $G$  in polynomial time in the size of  $G$ .
- ▶ This implies that every minor-hereditary property can be tested in polynomial time.
- ▶ However, the constant factors involved in the known general algorithms for testing minors (which depend on  $|H|$ ) are so astronomically huge as to make them infeasible in practice.