

Discrete Mathematics in Computer Science

Graphs and Directed Graphs

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Graphs

Graphs (of various kinds) are ubiquitous in Computer Science and its applications.

Some examples:

- Boolean circuits in hardware design
- control flow graphs in compilers
- pathfinding in video games
- computer networks
- neural networks
- social networks

Graph Theory

- **Graph theory** was founded in 1736 by Leonhard Euler's study of the **Seven Bridges of Königsberg** problem.
- It remains one of the main areas of discrete mathematics to this day.

More on Euler and the Seven Bridges of Königsberg:



- The Seven Bridges of Königsberg – Numberphile.
<https://youtu.be/W18FDEA1jRQ>

Graphs and Directed Graphs – Definitions

Definition (Graph)

A **graph** (also: **undirected graph**) is a pair $G = (V, E)$, where

- V is a finite set called the set of **vertices**, and
- $E \subseteq \{\{u, v\} \subseteq V \mid u \neq v\}$ is called the set of **edges**.

German: Graph, ungerichteter Graph, Knoten, Kanten

Graphs and Directed Graphs – Definitions

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Definition (Directed Graph)

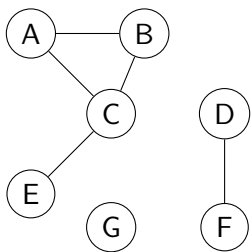
A **directed graph** (also: **digraph**) is a pair $G = (N, A)$, where

- N is a finite set called the set of **nodes**, and
- $A \subseteq N \times N$ is called the set of **arcs**.

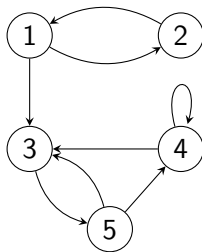
German: gerichteter Graph, Digraph, Knoten, Kanten/Pfeile

Graphs and Directed Graphs – Pictorially

often described pictorially:



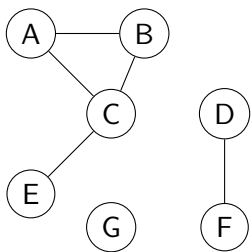
graph (V, E)



directed graph (N, A)

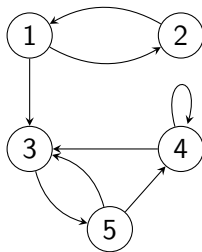
Graphs and Directed Graphs – Pictorially

often described pictorially:



graph (V, E)

- $V = \{A, B, C, D, E, F, G\}$
- $E = \{\{A, B\}, \{A, C\}, \{B, C\}, \{C, E\}, \{D, F\}\}$



directed graph (N, A)

- $N = \{1, 2, 3, 4, 5\}$
- $A = \{(1, 2), (1, 3), (2, 1), (3, 5), (4, 3), (4, 4), (5, 3), (5, 4)\}$

Relationship to Relations

graphs vs. directed graphs:

- edges are **sets** of two elements, arcs are **pairs**
- arcs can be **self-loops** (v, v) ; edges cannot (**why not?**)

(di-)graphs vs. relations:

- A directed graph (N, A) is essentially identical to
(= contains the same information as)
an **arbitrary relation** R_A over the finite set N :
 $u R_A v$ iff $(u, v) \in A$
- A graph (V, E) is essentially identical to
an **irreflexive symmetric** relation R_E over the finite set V :
 $u R_E v$ iff $\{u, v\} \in E$

Other Kinds of Graphs

many variations exist, for example:

- self-loops may be allowed in edges (“non-simple” graphs)
- labeled graphs: additional information associated with vertices and/or edges
- weighted graphs: numbers associated with edges
- multigraphs: multiple edges between same vertices allowed
- mixed graphs: both edges and arcs allowed
- hypergraphs: edges can involve more than 2 vertices
- infinite graphs: may have infinitely many vertices/edges

Graph Terminology

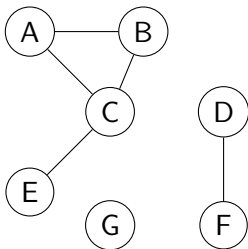
Definition (Graph Terminology)

Let (V, E) be a graph.

- u and v are the **endpoints** of the edge $\{u, v\} \in E$
- u and v are **incident** to the edge $\{u, v\} \in E$
- u and v are **adjacent** if $\{u, v\} \in E$
- the vertices adjacent with $v \in V$ are its **neighbours** $\text{neigh}(v)$:
 $\text{neigh}(v) = \{w \in V \mid \{v, w\} \in E\}$
- the number of neighbours of $v \in V$ is its **degree** $\text{deg}(v)$:
 $\text{deg}(v) = |\text{neigh}(v)|$

German: Endknoten, inzident, adjazent/benachbart, Nachbarn, Grad

Graph Terminology – Examples



endpoints, incident, adjacent, neighbours, degree

Directed Graph Terminology

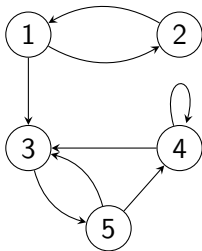
Definition (Directed Graph Terminology)

Let (N, A) be a directed graph.

- u is the **tail** and v is the **head** of the arc $(u, v) \in A$;
we say (u, v) is an arc **from** u **to** v
- u and v are **incident** to the arc $(u, v) \in A$
- u is a **predecessor** of v and v is a **successor** of u if $(u, v) \in A$
- the predecessors and successor of v are written as
pred $(v) = \{u \in N \mid (u, v) \in A\}$ and
succ $(v) = \{w \in N \mid (v, w) \in A\}$
- the number of predecessors/successors of $v \in N$ is its
indegree/outdegree: $\text{indeg}(v) = |\text{pred}(v)|$,
 $\text{outdeg}(v) = |\text{succ}(v)|$

German: Fuss, Kopf, inzident, Vorgänger, Nachfolger,
Eingangs-/Ausgangsgrad

Directed Graph Terminology – Examples



head, tail, predecessors, successors, indegree, outdegree

Discrete Mathematics in Computer Science

Induced Graphs and Degree Lemma

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Induced Graph of a Directed Graph

Definition (undirected graph induced by a directed graph)

Let $G = (N, A)$ be a directed graph.

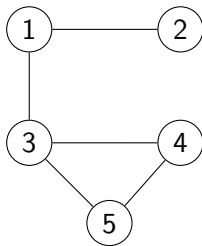
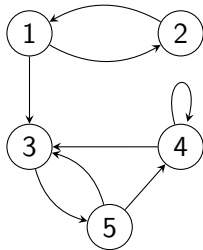
The (undirected) **graph induced by G** is the graph (N, E) with $E = \{\{u, v\} \mid (u, v) \in A, u \neq v\}$.

German: induziert

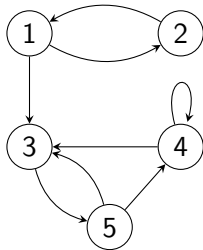
Questions:

- Why require $u \neq v$?
- If $|N| = n$ and $|A| = m$, how many vertices and edges does the induced graph have?
- How does the answer change if G has no self-loops?

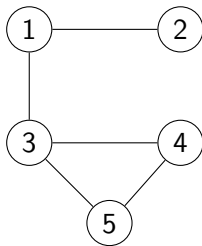
Induced Graph of a Directed Graph – Example



Induced Graph of a Directed Graph – Example



- $N = \{1, 2, 3, 4, 5\}$
- $A = \{(1, 2), (1, 3), (2, 1), (3, 5), (4, 3), (4, 4), (5, 3), (5, 4)\}$



- $V = \{1, 2, 3, 4, 5\}$
- $E = \{\{1, 2\}, \{1, 3\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\}$

Degree Lemma

Lemma (degree lemma for directed graphs)

Let (N, A) be a directed graph.

Then $\sum_{v \in N} \text{indeg}(v) = \sum_{v \in N} \text{outdeg}(v) = |A|$.

Intuitively: every arc contributes 1 to the indegree of one node and 1 to the outdegree of one node.

Degree Lemma

Lemma (degree lemma for directed graphs)

Let (N, A) be a directed graph.

Then $\sum_{v \in N} \text{indeg}(v) = \sum_{v \in N} \text{outdeg}(v) = |A|$.

Intuitively: every arc contributes 1 to the indegree of one node and 1 to the outdegree of one node.

Lemma (degree lemma for undirected graphs)

Let (V, E) be a graph.

Then $\sum_{v \in V} \text{deg}(v) = 2|E|$.

Intuitively: every edge contributes 1 to the degree of two vertices.

Degree Lemma

Lemma (degree lemma for directed graphs)

Let (N, A) be a directed graph.

Then $\sum_{v \in N} \text{indeg}(v) = \sum_{v \in N} \text{outdeg}(v) = |A|$.

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Lemma (degree lemma for undirected graphs)

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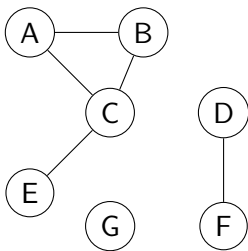
Then $\sum_{v \in V} \text{deg}(v) = 2|E|$.

Intuitively: every edge contributes 1 to the degree of two vertices.

Corollary

Every graph has an even number of vertices with odd degree.

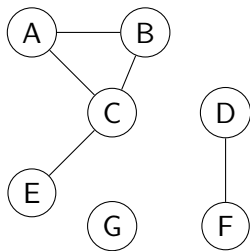
Degree Lemma – Example



$$\sum_{v \in V} \deg(v)$$

$$= \deg(A) + \deg(B) + \deg(C) + \deg(D) + \deg(E) + \deg(F) + \deg(G)$$

Degree Lemma – Example

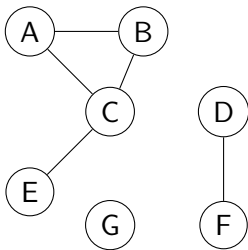


$$\sum_{v \in V} \deg(v)$$

$$= \deg(A) + \deg(B) + \deg(C) + \deg(D) + \deg(E) + \deg(F) + \deg(G)$$

$$= 2 + 2 + 3 + 1 + 1 + 1 + 0$$

Degree Lemma – Example



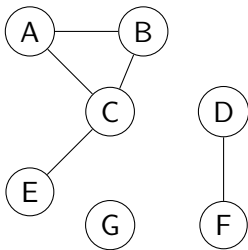
$$\sum_{v \in V} \deg(v)$$

$$= \deg(A) + \deg(B) + \deg(C) + \deg(D) + \deg(E) + \deg(F) + \deg(G)$$

$$= 2 + 2 + 3 + 1 + 1 + 1 + 0$$

$$= 10 = 2 \cdot 5 = 2|E|$$

Degree Lemma – Example



$$\sum_{v \in V} \deg(v)$$

$$= \deg(A) + \deg(B) + \deg(C) + \deg(D) + \deg(E) + \deg(F) + \deg(G)$$

$$= 2 + 2 + 3 + 1 + 1 + 1 + 0$$

$$= 10 = 2 \cdot 5 = 2|E|$$

4 vertices with odd degree

Degree Lemma – Proof (1)

Proof of degree lemma for directed graphs.

$$\begin{aligned}\sum_{v \in N} \text{indeg}(v) &= \sum_{v \in N} |\text{pred}(v)| \\&= \sum_{v \in N} |\{u \mid u \in N, (u, v) \in A\}| \\&= \sum_{v \in N} |\{(u, v) \mid u \in N, (u, v) \in A\}| \\&= \left| \bigcup_{v \in N} \{(u, v) \mid u \in N, (u, v) \in A\} \right| \\&= |\{(u, v) \mid u \in N, v \in N, (u, v) \in A\}| \\&= |A|.\end{aligned}$$

$\sum_{v \in N} \text{outdeg}(v) = |A|$ is analogous.



Degree Lemma – Proof (2)

We omit the proof for undirected graphs, which can be conducted similarly.

One possible proof strategy that reuses the result we proved:

- Define **directed** graph (V, A) from the graph (V, E) by orienting each edge into an arc arbitrarily.
- Observe $\deg(v) = \text{indeg}(v) + \text{outdeg}(v)$, where \deg refers to the graph and $\text{indeg}/\text{outdeg}$ to the directed graph.
- Use the degree lemma for directed graphs:
$$\sum_{v \in V} \deg(v) = \sum_{v \in V} (\text{indeg}(v) + \text{outdeg}(v)) =$$
$$\sum_{v \in V} \text{indeg}(v) + \sum_{v \in V} \text{outdeg}(v) = |A| + |A| = 2|A| = 2|E|$$