# Discrete Mathematics in Computer Science

## B10. A Glimpse of Abstract Algebra

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# B10.1 Abstract Groups

# Abstract Algebra

- Elementary algebra: "Arithmetics with variables"
  - e. g. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ describes the solutions of $ax^2 + bx + c = 0$ where $a \neq 0$.
  - Variables for numbers and operations such as addition, subtraction, multiplication, division ...
  - "What you learn at school."
- Abstract algebra: Generalization of elementary algebra
  - Arbitrary sets and operations on their elements
  - e. g. permutations of a given set $S$ plus function composition
  - Abstract algebra studies arbitrary sets and operations based on certain properties (such as associativity).

# Binary operations

- A binary operation on a set $S$ is a function $f : S \times S \to S$.
- e. g. $add : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$ for addition of natural numbers.
- In infix notation, we write the operator between the operands, e. g. $x + y$ instead of $add(x, y)$.

# Groups

## Definition (Group)

A group $G = (S, \cdot)$ is given by a set $S$ and
a binary operation $\cdot$ on $S$ that satisfy the group axioms:

- Associativity: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in S$.
- Identity element: There exists an $e \in S$ such that for all $x \in S$ it holds that $x \cdot e = e \cdot x = x$.
  Element $e$ is called identity of neutral element of the group.
- Inverse element: For every $x \in S$ there is a $y \in S$ such that $x \cdot y = y \cdot x = e$, where $e$ is the identity element.

A group is called abelian if $\cdot$ is also commutative,
i. e. for all $x, y \in S$ it holds that $x \cdot y = y \cdot x$.

Cardinality $|S|$ is called the order of the group.

Niels Henrik Abel: Norwegian mathematician (1802–1829),
cf. Abel prize

# Example: $(\mathbb{Z}, +)$

$(\mathbb{Z}, +)$ is a group:

- $\mathbb{Z}$ is closed under addition, i. e. for $x, y \in \mathbb{Z}$ it holds that $x + y \in \mathbb{Z}$
- The $+$ operator is associative: for all $x, x, z \in \mathbb{Z}$ it holds that $(x + y) + z = x + (y + z)$.
- Integer 0 is the neutral element: for all integers $x$ it holds that $x + 0 = 0 + x = x$.
- Every integer $x$ has an inverse element in the integers, namely $-x$, because $x + (-x) = (-x) + x = 0$.

$(\mathbb{Z}, +)$ also is an abelian group
because for all $x, y \in \mathbb{Z}$ it holds that $x + y = y + x$.

# Uniqueness of Identity and Inverses

**Theorem**

*Every group $G = (S, \cdot)$ has only one identity element and for each $x \in S$ the inverse of $x$ is unique.*

**Proof.**

identity: Assume that there are two identity elements $e, e' \in S$ with $e \neq e'$. Then for all $x \in S$ it holds that $x \cdot e = e \cdot x = x$ and that $x \cdot e' = e' \cdot x = x$. Using $x = e'$, we get $e' \cdot e = e'$ and using $x = e$ we get $e' \cdot e = e$, so overall $e' = e$. $\lightning$

inverse: homework assignment □

We often denote the identity element with $\mathbf{1}$ and the inverse of $x$ with $x^{-1}$.

# Division – Right Quotient

**Theorem**

*Let $G = (S, \cdot)$ be a group. Then for all $a, b \in S$ the equation $x \cdot b = a$ has exactly one solution $x$ in $S$, namely $x = a \cdot b^{-1}$.*

*We call $a \cdot b^{-1}$ the right-quotient of $a$ by $b$ and also write it as $a/b$.*

**Proof.**

It is a solution: With $x = a \cdot b^{-1}$ it holds that
$x \cdot b = (a \cdot b^{-1}) \cdot b = a \cdot (b^{-1} \cdot b) = a \cdot \mathbf{1} = a$.

The solution is unique:
Assume $x$ and $x'$ are distinct solutions. Then $x \cdot b = a = x' \cdot b$.
Multiplying both sides by $b^{-1}$, we get $(x \cdot b) \cdot b^{-1} = (x' \cdot b) \cdot b^{-1}$ and with associativity $x \cdot (b \cdot b^{-1}) = x' \cdot (b \cdot b^{-1})$.
With the axiom on inverse elements this leads to $x \cdot \mathbf{1} = x' \cdot \mathbf{1}$ and with the axiom on the identity element ultimately to $x = x'$. $\lightning$ □

# Division – Left Quotient

**Theorem**

*Let $G = (S, \cdot)$ be a group. Then for all $a, b \in S$ the equation $b \cdot x = a$ has exactly one solution $x$ in $S$, namely $x = b^{-1} \cdot a$.*

*We call $b^{-1} \cdot a$ the left-quotient of $a$ by $b$ and also write it as $b \setminus a$.*

Proof omitted

# Quotients in Abelian Groups

**Theorem**

*If $G = (S, \cdot)$ is an abelian group then it holds for all $x, y \in S$ that $x/y = y\backslash x$.*

**Proof.**

Consider arbitrary $x, y \in S$. As $\cdot$ is commutative, it holds that $x/y = x \cdot y^{-1} = y^{-1} \cdot x = y\backslash x$. □

# Group Homomorphism

A group homomorphism is a function that preserves group structure:

**Definition (Group homomorphism)**

Let $G = (S, \cdot)$ and $G' = (S', \circ)$ be groups.
A homomorphism from $G$ to $G'$ is a function $f : S \to S'$ such that for all $x, y \in S$ it holds that $f(x \cdot y) = f(x) \circ f(y)$.

**Definition (Group Isomorphism)**

A group homomorphism that is bijective is called a group isomorophism. Groups $G$ and $H$ are called isomorphic if there is a group isomorphism from $G$ to $H$.

From a practical perspective, isomorphic groups are identical up to renaming.

# Group Homomorphism – Example

- Consider $G = (\mathbb{Z}, +)$ and $H = (\{1, -1\}, \cdot)$ with
  - $1 \cdot 1 = -1 \cdot -1 = 1$
  - $1 \cdot -1 = -1 \cdot 1 = -1$
- Let $f : \mathbb{Z} \to \{1, -1\}$ with $f(x) = \begin{cases} 1 & \text{if } x \text{ is even} \\ -1 & \text{if } x \text{ is odd} \end{cases}$
- $f$ is a homomorphism from $G$ to $H$:
  for all $x, y \in \mathbb{Z}$ it holds that

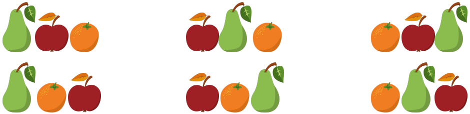
$$
\begin{aligned}
f(x+y) &= \begin{cases} 1 & \text{if } x+y \text{ is even} \\ -1 & \text{if } x+y \text{ is odd} \end{cases} \\
&= \begin{cases} 1 & \text{if } x \text{ and } y \text{ have the same parity} \\ -1 & \text{if } x \text{ and } y \text{ have different parity} \end{cases} \\
&= \begin{cases} 1 & \text{if } f(x) = f(y) \\ -1 & \text{if } f(x) \neq f(y) \end{cases} \\
&= f(x) \cdot f(y)
\end{aligned}
$$

# Outlook

- A subgroup of $G = (S, \cdot)$ is a group $H = (S', \circ)$ with $S' \subseteq S$ and $\circ$ the restriction of $\cdot$ to $S' \times S'$.
  - $S'$ always contains the identity element and is closed under group operation and inverse
  - group homomorphisms preserve many properties of subgroups
- Other algebraic structures, e. g.
  - Semi-group: requires only associativity
  - Monoid: requires associativity and identity element
  - Ringoids: algebraic structures with two binary operations
    - multiplication and addition
    - multiplication distributes over addition
    - e. g. ring and field

# B10.2 Symmetric Group and Permutation Groups

# Reminder: Permutations



**Definition (Permutation)**

Let $S$ be a set. A bijection $\pi : S \to S$ is called a permutation of $S$.

# Symmetric Group

**Theorem (Symmetric Group)**

*Let $M$ be a set. Then* $\mathrm{Sym}(M) = (S, \cdot)$*, where*

- *$S$ is the set of all permutations of $M$, and*
- *$\cdot$ denotes function composition,*

*is a group, called the symmetric group of $M$.*

*For finite set $M = \{1, \dots, n\}$, we also use $S_n$ to refer to the symmetric group of $M$.*

Is the symmetric group abelian?

What's the order of $S_n$?

# Symmetric Group – Proof I

**Theorem**

*For set $M$, $\mathrm{Sym}(M) = (\{\sigma : M \to M \mid \sigma \text{ is bijective}\}, \cdot)$ is a group.*

**Definition (Group)**

A group $G = (S, \cdot)$ is given by a set $S$ and
a binary operation $\cdot$ on $S$ that satisfy the group axioms:

- Associativity: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in S$.
- Identity element: There exists an $e \in S$ such that for all $x \in S$ it holds that $x \cdot e = e \cdot x = x$.
  Element $e$ is called identity of neutral element of the group.
- Inverse element: For every $x \in S$ there is a $y \in S$ such that $x \cdot y = y \cdot x = e$, where $e$ is the identity element.

To show: closure, associativity, identity, inverse element

# Symmetric Group – Proof II

**Theorem**

*For set $M$, $\mathrm{Sym}(M) = (\{\sigma : M \to M \mid \sigma \text{ is bijective}\}, \cdot)$ is a group.*

**Proof.**

- Closure: The product of two permutations of $M$ is a permutation of $M$ and hence in the set.
- Associativity: Function composition is always associative.
- Identity element: Function $\mathrm{id} : M \to M$ with $\mathrm{id}(x) = x$ is a permutation and for every permutation $\sigma$ of $M$ it holds that $\sigma\mathrm{id} = \mathrm{id}\sigma = \sigma$.
- Inverse element: For every permutation $\sigma$ of $M$, also the inverse function $\sigma^{-1}$ is a permutation of $M$ and has the required properties.

□

# Generating Sets

**Definition**

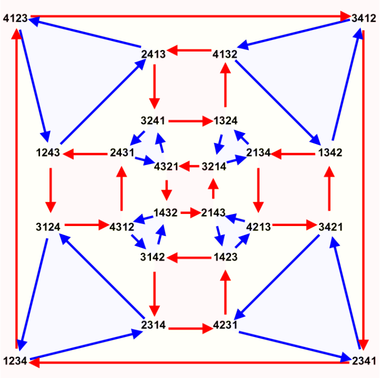
A generating set of a group $G = (S, \circ)$ is a set $S' \subseteq S$ such that every $e \in S$ can be expressed as a combination (under $\circ$) of finitely many elements of $S'$ and their inverses.

Empty product is identity by definition, so no need to have it in $S'$.

- For $n \geq 2$, $S_n$ is generated by $\{(i \quad i+1) \mid i \in \{1, \dots, n-1\}\}$.
- For $n > 2$, $S_n$ is generated by $\{(1 \quad 2), (1 \quad \dots \quad n)\}$.

# Generating Sets – Example

$$\left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} \right\}$$ is a generating set of $S_4$.



# Permutation Group

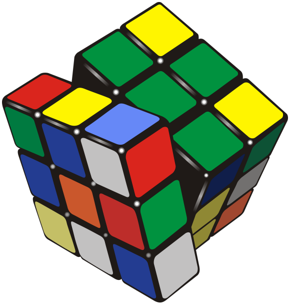
Sometimes, we do not want to consider all possible permutations.

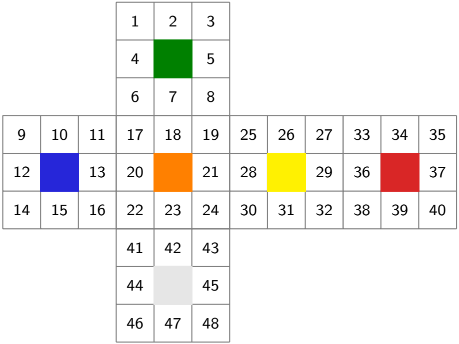
**Definition (Permutation Group)**

A permutation group is a group $G = (S, \cdot)$,
where $S$ is a set of permutations of some set $M$ and
$\cdot$ is the composition of permutations in $S$.

Every permutation group is a subgroup of a symmetric group and every such subgroup is a permutation group.

# Permutation Group – Example






- Consider all permutations achievable with valid moves.
- Subgroup of $S_{48}$ with order $43\,252\,003\,274\,489\,856\,000 \approx 4.3 \cdot 10^{19}$ (43 quintillion)
- $S_{48}$ has order $48! \approx 1.24 \cdot 10^{61}$