Discrete Mathematics in Computer Science

Permutations

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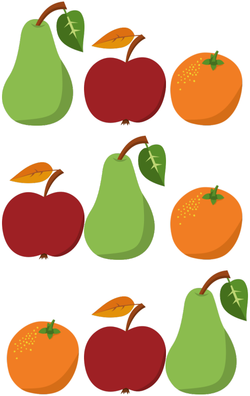

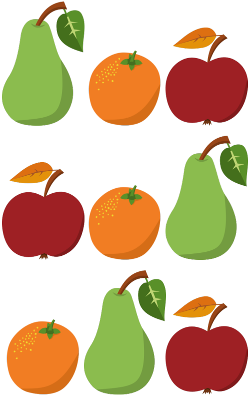

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- We call such a bijection a permutation.

Permutation – Definition

Definition (Permutation)

Let S be a set. A bijection $\pi : S \to S$ is called a permutation of S.

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How many permutations are there for a finite set S?

Two-line and One-line Notation (for Finite Sets)

Consider π with
$\pi(1) = 2, \pi(2) = 5, \pi(3) = 4, \pi(4) = 3, \pi(5) = 1, \pi(6) = 6.$

Two-line notation lists the elements of S in the first row and the image of each element in the second row:

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 4 & 3 & 1 & 6 \end{pmatrix} = \begin{pmatrix} 3 & 5 & 1 & 6 & 4 & 2 \\ 4 & 1 & 2 & 6 & 3 & 5 \end{pmatrix}$$

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One-line notation only lists the second row for the natural order of the first row:

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- The product of permutations is a permutation. Why?
- Example:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 4 & 1 & 5 \end{pmatrix} \qquad \pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 5 & 2 & 4 \end{pmatrix}$$

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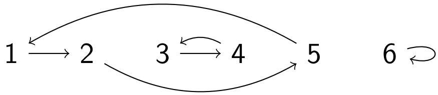

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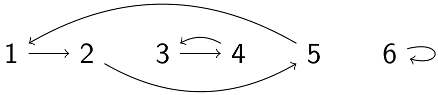

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and a cycle $(3 \quad 4) = (4 \quad 3)$.

Idea: Write π as product of such cycles.

Cycles

Definition (Cycle)

A permutation σ of finite set S has a *k-cycle* $(e_1 \quad e_2 \quad \dots \quad e_k)$ if

- $e_i \in S$ for $i \in \{1, \dots, k\}$
- $e_i \neq e_j$ for $i \neq j$
- $\sigma(e_i) = e_{i+1}$ for $i \in \{1, \dots, k-1\}$
- $\sigma(e_k) = e_1$

- Don't confuse cycles with permutations in one-line notation.
- A 2-cycle is called a transposition
- A 1-cycle is called a fixed-point of σ.

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Question: Is this representation unique (canonical)?

Cycle Notation – Example

We can write every permutation as a product of disjoint cycles.

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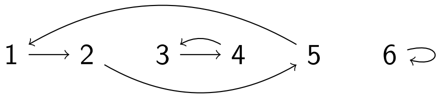

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In cycle representation:
$\pi = (1 \quad 2 \quad 5)(3 \quad 4)(6) = (1 \quad 2 \quad 5)(3 \quad 4)$

Cycle Notation – Algorithm

Let π be a permutation of finite set S.

```
function ComputeCycleRepresentation(π, S)
    remaining = S
    cycles = ∅
    while remaining is not empty do
        Remove any element e from remaining.
        Start a new cycle c with e.
        while π(e) ∈ remaining do
            remaining = remaining \ {π(e)}
            Extend c with π(e).
            e = π(e)
        cycles = cycles ∪ {c}
    return cycles
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Disjoint Cycles Commute

Theorem

Let $\pi = (e_1 \quad \ldots \quad e_n)$ *and* $\pi' = (e'_1 \quad \ldots \quad e'_m)$ *be permutations of set* S *in cycle notation and let* π *and* π' *be disjoint, i. e.* $e_i \neq e'_j$ *for* $i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\}$.

Then $\pi\pi' = \pi'\pi$.

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If e occurs in neither cycle then $\pi(e) = e$ and $\pi'(e) = e$, so $\pi'(\pi(e)) = e = \pi(\pi'(e))$. □

In General Cycles Do not Commute

Consider cycles $(1 \quad 2)$ and $(2 \quad 3)$ and set $S = \{1, 2, 3\}$.

$(1 \quad 2)(2 \quad 3) =$

$(2 \quad 3)(1 \quad 2) =$

Transpositions

Theorem

Every cycle can be expressed as a product of transpositions.

Proof idea.

Consider k-cycle $\sigma = (e_1 \quad \dots \quad e_k)$.
We can express σ as $(e_1 \quad e_k)(e_1 \quad e_{k-1})\dots(e_1 \quad e_2)$. □

Inverse

- Every permutation has an inverse, which is again a permuation.
  - If π is represented in two-line notation, we get π^{-1} by swapping the rows, e. g.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 4 & 1 & 5 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 2 & 4 & 1 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

  - If π is a cycle, we get π^{-1} by reversing the order of the elements, e. g. $(1 \quad 3 \quad 4 \quad 2)^{-1} = (2 \quad 4 \quad 3 \quad 1)$
  - $(\pi\sigma)^{-1} = \sigma^{-1}\pi^{-1}$

Example

$\sigma = (4 \quad 5)(2 \quad 3) \qquad \pi = (4 \quad 5)(2 \quad 1)$

$\sigma\pi^{-1} =$

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Then $\pi \circ f$ describes the resulting configuration.

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Then $g \circ f^{-1}$ describes the permutation.