Discrete Mathematics in Computer Science

Operations on Relations

Malte Helmert, Gabriele Röger

University of Basel

Relations: Recap

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- A homogeneous relation R over set S is a binary relation $R \subseteq S \times S$.

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  With the standard relations for $\mathbb{N}_0$, relation $=$ is the complementary relation of $\neq$ and $>$ the one of $\leq$.

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Let $R \subseteq A \times B$ be a binary relation over A and B.

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- The inverse of the $<$ relation over $\mathbb{N}_0$ is the $>$ relation.
- Relation R with xRy iff person x has a key for y.
  Inverse: Q with aQb iff lock a can be opened by person b.

Composition of Relations

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Let R_1 be a relation over A and B and
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The composition of R_1 and R_2 is the relation $R_2 \circ R_1$ with:

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How can we illustrate this graphically?

Composition is Associative

Theorem (Associativity of composition)

Let $S_1, \ldots, S_4$ be sets and R_1, R_2, R_3 relations with $R_i \subseteq S_i \times S_{i+1}$. Then

$$R_3 \circ (R_2 \circ R_1) = (R_3 \circ R_2) \circ R_1.$$

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Proof.

It holds that $(x_1, x_4) \in R_3 \circ (R_2 \circ R_1)$ iff there is an x_3 with $(x_1, x_3) \in R_2 \circ R_1$ and $(x_3, x_4) \in R_3$.

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As $(x_1, x_3) \in R_2 \circ R_1$ iff there is an x_2 with $(x_1, x_2) \in R_1$ and $(x_2, x_3) \in R_2$, we have overall that $(x_1, x_4) \in R_3 \circ (R_2 \circ R_1)$ iff there are x_2, x_3 with $(x_1, x_2) \in R_1$, $(x_2, x_3) \in R_2$ and $(x_3, x_4) \in R_3$.

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This is the case iff there is an x_2 with $(x_1, x_2) \in R_1$ and $(x_2, x_4) \in R_3 \circ R_2$, which holds iff $(x_1, x_4) \in (R_3 \circ R_2) \circ R_1$. $\square$

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Example: If aRb specifies that block a lies on block b, what does R^* express?

Transitive Closure II

Define the i-th power of a homogeneous relation R as

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Theorem

Let R be a relation over set S. Then $R^* = \bigcup_{i=1}^{\infty} R^i$.

Without proof.

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- For example, join operators combine relations based on common entries.
- Example for a natural join:

Employee

Name	EmpId	DeptName
Harry	3415	Finance
Sally	2241	Sales
George	3401	Finance
Harriet	2202	Sales
Mary	1257	Human Resources

Dept

DeptName	Manager
Finance	George
Sales	Harriet
Production	Charles

Employee ⋈ Dept

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Harry	3415	Finance	George
Sally	2241	Sales	Harriet
George	3401	Finance	George
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(Source: Wikipedia)

Summary

- Relations: general, binary, homogeneous
- Properties: reflexivity, symmetry, transitivity
  (and related properties)
- Special relations: equivalence relations, order relations
- Operations: inverse, composition, transitive closure