

Discrete Mathematics in Computer Science

B7. Operations on Relations

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Relations: Recap

- ▶ A **relation over sets** S_1, \dots, S_n is a set $R \subseteq S_1 \times \dots \times S_n$.
- ▶ A **binary** relation is a relation over two sets.
- ▶ A **homogeneous** relation R over set S is a binary relation $R \subseteq S \times S$.

Set Operations

- ▶ Relations are sets of tuples, so we can build their union, intersection, complement,
- ▶ Let R be a relation over S_1, \dots, S_n and R' a relation over S'_1, \dots, S'_n . Then $R \cup R'$ is a relation over $S_1 \cup S'_1, \dots, S_n \cup S'_n$.
With the standard relations $<, =$ and \leq for \mathbb{N}_0 ,
relation \leq corresponds to the union of relations $<$ and $=$.
- ▶ Let R and R' be relations over n sets.
Then $R \cap R'$ is a relation.
Over which sets?
With the standard relations $\leq, =$ and \geq for \mathbb{N}_0 ,
relation $=$ corresponds to the intersection of \leq and \geq .
- ▶ If R is a relation over S_1, \dots, S_n
then so is the **complementary relation** $\bar{R} = (S_1 \times \dots \times S_n) \setminus R$.
With the standard relations for \mathbb{N}_0 , relation $=$ is the
complementary relation of \neq and $>$ the one of \leq .

Inverse of a Relation

Definition

Let $R \subseteq A \times B$ be a binary relation over A and B .

The **inverse relation** of R is the relation $R^{-1} \subseteq B \times A$ given by $R^{-1} = \{(b, a) \mid (a, b) \in R\}$.

- ▶ The inverse of the $<$ relation over \mathbb{N}_0 is the $>$ relation.
- ▶ Relation R with xRy iff person x has a key for y .
Inverse: Q with aQb iff lock a can be opened by person b .

Composition of Relations

Definition (Composition of relations)

Let R_1 be a relation over A and B and R_2 be a relation over B and C .

The **composition of R_1 and R_2** is the relation $R_2 \circ R_1$ with:

$$R_2 \circ R_1 = \{(a, c) \mid \text{there is a } b \in B \text{ with} \\ (a, b) \in R_1 \text{ and } (b, c) \in R_2\}$$

How can we illustrate this graphically?

Composition is Associative

Theorem (Associativity of composition)

Let S_1, \dots, S_4 be sets and R_1, R_2, R_3 relations with $R_i \subseteq S_i \times S_{i+1}$.
Then

$$R_3 \circ (R_2 \circ R_1) = (R_3 \circ R_2) \circ R_1.$$

Proof.

It holds that $(x_1, x_4) \in R_3 \circ (R_2 \circ R_1)$ iff there is an x_3 with $(x_1, x_3) \in R_2 \circ R_1$ and $(x_3, x_4) \in R_3$.

As $(x_1, x_3) \in R_2 \circ R_1$ iff there is an x_2 with $(x_1, x_2) \in R_1$ and $(x_2, x_3) \in R_2$, we have overall that $(x_1, x_4) \in R_3 \circ (R_2 \circ R_1)$ iff there are x_2, x_3 with $(x_1, x_2) \in R_1$, $(x_2, x_3) \in R_2$ and $(x_3, x_4) \in R_3$.

This is the case iff there is an x_2 with $(x_1, x_2) \in R_1$ and $(x_2, x_4) \in R_3 \circ R_2$, which holds iff $(x_1, x_4) \in (R_3 \circ R_2) \circ R_1$. \square

Transitive Closure

Definition (Transitive closure)

The **transitive closure** R^* of a relation R over set S is the **smallest relation over S that is transitive and has R as a subset.**

The transitive closure always exists. *Why?*

Example: If aRb specifies that block a lies on block b ,
what does R^* express?

Transitive Closure II

Define the i -th power of a homogeneous relation R as

$$\begin{aligned} R^1 &= R && \text{if } i = 1 \text{ and} \\ R^i &= R \circ R^{i-1} && \text{for } i > 1 \end{aligned}$$

Theorem

Let R be a relation over set S . Then $R^* = \bigcup_{i=1}^{\infty} R^i$.

Without proof.

Other Operators

- ▶ There are many more operators, also for general relations.
- ▶ Highly relevant for **queries over relational databases**.
- ▶ For example, **join** operators combine relations based on common entries.
- ▶ Example for a **natural join**:

Employee

Name	Empld	DeptName
Harry	3415	Finance
Sally	2241	Sales
George	3401	Finance
Harriet	2202	Sales
Mary	1257	Human Resources

Dept

DeptName	Manager
Finance	George
Sales	Harriet
Production	Charles

Employee ⋈ Dept

Name	Empld	DeptName	Manager
Harry	3415	Finance	George
Sally	2241	Sales	Harriet
George	3401	Finance	George
Harriet	2202	Sales	Harriet

(Source: Wikipedia)

Summary

- ▶ Relations: general, binary, homogeneous
- ▶ Properties: reflexivity, symmetry, transitivity (and related properties)
- ▶ Special relations: equivalence relations, order relations
- ▶ Operations: inverse, composition, transitive closure