

Discrete Mathematics in Computer Science

B6. Equivalence and Order Relations

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B6.1 Equivalence Relations and Partitions

B6.2 Partial Orders

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B6.1 Equivalence Relations and Partitions

Relations: Recap

- ▶ A **relation over sets** S_1, \dots, S_n is a set $R \subseteq S_1 \times \dots \times S_n$.
- ▶ Possible properties of homogeneous relations R over S :
 - ▶ **reflexive**: $(x, x) \in R$ for all $x \in S$
 - ▶ **irreflexive**: $(x, x) \notin R$ for all $x \in S$
 - ▶ **symmetric**: $(x, y) \in R$ iff $(y, x) \in R$
 - ▶ **asymmetric**: if $(x, y) \in R$ then $(y, x) \notin R$
 - ▶ **antisymmetric**: if $(x, y) \in R$ then $(y, x) \notin R$ or $x = y$
 - ▶ **transitive**: if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$

Motivation

- ▶ Think of any attribute that two objects can have in common, e. g. their color.
- ▶ We could place the objects into distinct “buckets”, e. g. one bucket for each color.
- ▶ We also can define a relation \sim such that $x \sim y$ iff x and y share the attribute, e. g. have the same color.
- ▶ Would this relation be
 - ▶ reflexive?
 - ▶ irreflexive?
 - ▶ symmetric?
 - ▶ asymmetric?
 - ▶ antisymmetric?
 - ▶ transitive?

Equivalence Relation

Definition (Equivalence Relation)

A binary relation \sim over set S is an **equivalence relation** if \sim is **reflexive, symmetric and transitive**.

Is this definition indeed what we want?

Does it allow us to partition the objects into buckets (e. g. one group for all objects that share a specific color)?

Partition

Definition (Partition)

A **partition** of a set S is a set $P \subseteq \mathcal{P}(S)$ such that

- ▶ $X \neq \emptyset$ for all $X \in P$,
- ▶ $\bigcup_{X \in P} X = S$, and
- ▶ $X \cap Y = \emptyset$ for all $X, Y \in P$ with $X \neq Y$,

The elements of P are called the **blocks** of the partition.

Partition

Let $S = \{e_1, \dots, e_5\}$.

Which of the following sets are partitions of S ?

- ▶ $P_1 = \{\{e_1, e_4\}, \{e_3\}, \{e_2, e_5\}\}$
- ▶ $P_2 = \{\{e_1, e_4, e_5\}, \{e_3\}\}$
- ▶ $P_3 = \{\{e_1, e_4, e_5\}, \{e_3\}, \{e_2, e_5\}\}$
- ▶ $P_4 = \{\{e_1\}, \{e_2\}, \{e_3\}, \{e_4\}, \{e_5\}\}$
- ▶ $P_5 = \{\{e_1\}, \{e_2\}, \{e_3\}, \{e_4\}, \{e_5\}, \{\}\}$

A Property of Partitions

Lemma

Let S be a set and P be a partition of S .

Then every $x \in S$ is an element of exactly one $X \in P$.

Proof: \rightsquigarrow exercises

Block of an Element

The lemma enables the following definition:

Definition

Let S be a set and P be a partition of S .

For $e \in S$ we denote by $[e]_P$ the block $X \in P$ such that $e \in X$.

Consider partition $P = \{\{e_1, e_4\}, \{e_3\}, \{e_2, e_5\}\}$ of $\{e_1, \dots, e_5\}$.

$[e_1]_P =$

Connection between Partitions and Equivalence Relations?

- ▶ We will now explore the connection between partitions and equivalence relations.
- ▶ **Spoiler:** They are essentially the same concept.

Partitions Induce Equivalence Relations I

Definition (Relation induced by a partition)

Let S be a set and P be a partition of S .

The **relation \sim_P induced by P** is the binary relation over S with

$$x \sim_P y \text{ iff } [x]_P = [y]_P.$$

$x \sim_P y$ iff x and y are in the same block of P .

Consider partition $P = \{\{1, 4, 5\}, \{2, 3\}\}$ of set $\{1, 2, \dots, 5\}$.

$$\sim_P = \{(1, 1), (1, 4), (1, 5), (4, 1), (4, 4), (4, 5), (5, 1), (5, 4), (5, 5), (2, 2), (2, 3), (3, 2), (3, 3)\}$$

We will show that \sim_P is an equivalence relation.

Partitions Induce Equivalence Relations II

Theorem

Let P be a partition of S .

Relation \sim_P induced by P is an **equivalence relation** over S .

Proof.

We need to show that \sim_P is reflexive, symmetric and transitive.

reflexive: As $=$ is reflexive it holds for all $x \in S$ that $[x]_P = [x]_P$ and hence also that $x \sim_P x$.

symmetric: If $x \sim_P y$ then $[x]_P = [y]_P$. With the symmetry of $=$ we get that $[y]_P = [x]_P$ and conclude that $y \sim_P x$.

transitive: If $x \sim_P y$ and $y \sim_P z$ then $[x]_P = [y]_P$ and $[y]_P = [z]_P$. As $=$ is transitive, it then also holds that $[x]_P = [z]_P$ and hence $x \sim_P z$. \square

Equivalence Classes

Definition (equivalence class)

Let R be an equivalence relation over set S .

For any $x \in S$, the **equivalence class of x** is the set

$$[x]_R = \{y \in S \mid xRy\}.$$

Consider

$$R = \{(1, 1), (1, 4), (1, 5), (4, 1), (4, 4), (4, 5), (5, 1), (5, 4), (5, 5), \\ (2, 2), (2, 3), (3, 2), (3, 3)\}$$

over set $\{1, 2, \dots, 5\}$.

$$[4]_R =$$

Equivalence Relations Induce Partitions

Theorem

Let R be an equivalence relation over set S .

The set $P = \{[x]_R \mid x \in S\}$ is a **partition of S** .

Proof.

We need to show that

- ① $X \neq \emptyset$ for all $X \in P$,
- ② $\bigcup_{X \in P} X = S$, and
- ③ $X \cap Y = \emptyset$ for all $X, Y \in P$ with $X \neq Y$,

1) For $x \in S$, it holds that $x \in [x]_R$ because R is reflexive.
Hence, no $X \in P$ is empty. \dots

Equivalence Relations Induce Partitions

Proof (continued).

For 2) we show $\bigcup_{X \in P} X \subseteq S$ and $\bigcup_{X \in P} X \supseteq S$ separately.

\subseteq : Consider an arbitrary $x \in \bigcup_{X \in P} X$. Since x is contained in the union, it must be an element of some $X \in P$. Consider such an X . By the definition of P , there is a $y \in S$ such that $X = [y]_R$.

Since $x \in [y]_R$, it holds that yRx .

As R is a relation over S , this implies that $x \in S$.

\supseteq : Consider an arbitrary $x \in S$. Since $x \in [x]_R$ (cf. 1) and $[x]_R \in P$, it holds that $x \in \bigcup_{X \in P} X$. \dots

Equivalence Relations Induce Partitions

Proof (continued).

We show 3) by contrapositive:

For all $X, Y \in P$: if $X \cap Y \neq \emptyset$ then $X = Y$.

Let X, Y be two sets from P with $X \cap Y \neq \emptyset$.

Then there is an e with $e \in X \cap Y$ and there are $x, y \in S$ with $X = [x]_R$ and $Y = [y]_R$. Consider such e, x, y .

As $e \in [x]_R$ and $e \in [y]_R$ it holds that xRe and yRe . Since R is symmetric, we get from yRe that eRy . By transitivity, xRe and eRy imply xRy , which by symmetry also gives yRx .

We show $[x]_R \subseteq [y]_R$: consider an arbitrary $z \in [x]_R$. Then xRz . From yRx and xRz , by transitivity we get yRz . This establishes $z \in [y]_R$. As z was chosen arbitrarily, it holds that $[x]_R \subseteq [y]_R$.

Analogously, we can show that $[x]_R \supseteq [y]_R$, so overall $X = Y$. \square

Summary

- ▶ We typically encounter equivalence relations when we consider objects as equivalent wrt. some attribute/property.
- ▶ A relation is an **equivalence relation** if it is **reflexive, symmetric and transitive**.
- ▶ A **partition** of a set groups the elements into non-empty subsets.
- ▶ The concepts are closely connected: in principle just different perspectives on the same "situation".

B6.2 Partial Orders

Order Relations

- ▶ An **equivalence relation** is **reflexive, symmetric and transitive**.
- ▶ Such a relation induces a partition into "equivalent" objects.
- ▶ We now consider **other combinations of properties**, that allow us to **compare objects** in a set against other objects.
- ▶ "Number x is not larger than number y ."
"Set S is a subset of set T ."
"Jerry runs at least as fast as Tom."
"Pasta tastes better than Potatoes."

Partial Orders

- ▶ We begin with **partial orders**.
- ▶ Example partial order relations are \leq over \mathbb{N} or \subseteq for sets.
- ▶ Are these relations
 - ▶ reflexive?
 - ▶ irreflexive?
 - ▶ symmetric?
 - ▶ asymmetric?
 - ▶ antisymmetric?
 - ▶ transitive?

Partial Orders – Definition

Definition (Partial order, partially ordered sets)

A binary relation \preceq over set S is a **partial order** if \preceq is **reflexive, antisymmetric and transitive**.

A **partially ordered set** (or **poset**) is a pair (S, R) where S is a **set** and R is a **partial order over S** .

Which of these relations are partial orders?

- ▶ strict subset relation \subset for sets
- ▶ not-less-than relation \geq over \mathbb{N}_0
- ▶ $R = \{(a, a), (a, b), (b, b), (b, c), (c, c)\}$ over $\{a, b, c\}$

Least and Greatest Element

Some special elements of posets:

Definition (Least and greatest element)

Let \preceq be a partial order over set S .

An element $x \in S$ is the **least element** of S if **for all $y \in S$ it holds that $x \preceq y$** .

It is the **greatest element** of S if **for all $y \in S, y \preceq x$** .

- ▶ Is there a least/greatest element? Which one?
 - ▶ $S = \{1, 2, 3\}$ and $\preceq = \{(x, y) \mid x, y \in S \text{ and } x \leq y\}$.
 - ▶ \mathbb{N}_0 and standard relation \leq .
- ▶ Why can we say **the** least element instead of **a** least element?

Uniqueness of Least Element

Theorem

Let \preceq be a partial order over set S .

If S contains a least element, it contains exactly one least element.

Proof.

By contradiction: Assume x, y are least elements of S with $x \neq y$.

Since x is a least element, $x \preceq y$ is true.

Since y is a least element, $y \preceq x$ is true.

As a partial order is antisymmetric, this implies that $x = y$. ζ \square

Analogously: If there is a greatest element then is unique.

Minimal and Maximal Elements

Definition (Minimal/Maximal element of a set)

Let \preceq be a partial order over set S .

An element $x \in S$ is a **minimal element** of S
if **there is no** $y \in S$ with $y \preceq x$ and $x \neq y$.

An element $x \in S$ is a **maximal element** of S
if **there is no** $y \in S$ with $x \preceq y$ and $x \neq y$.

A set can have several minimal elements and no least element.

Example?

Total Orders

- ▶ Relations \leq over \mathbb{N}_0 and \subseteq for sets are partial orders.
- ▶ Can we compare every object against every object?
 - ▶ For all $x, y \in \mathbb{N}_0$ it holds that $x \leq y$ or that $y \leq x$ (or both).
 - ▶ $\{1, 2\} \not\subseteq \{2, 3\}$ and $\{2, 3\} \not\subseteq \{1, 2\}$
- ▶ Relation \leq is a **total** order, relation \subseteq is not.

Total Order – Definition

Definition (Total relation)

A binary relation R over set S is **total** (or **connex**)
if for all $x, y \in S$ at least one of xRy or yRx is true.

Definition (Total order)

A binary relation is a **total order** if it is **total** and a **partial order**.

Summary

- ▶ A **partial order** is **reflexive, antisymmetric and transitive**.
- ▶ With a **total order** \preceq over S there are no elements $x, y \in S$ with $x \not\preceq y$ and $y \not\preceq x$.
- ▶ If x is the **greatest element** of a set S , it is greater than every element: for all $y \in S$ it holds that $y \preceq x$.
- ▶ If x is a **maximal element** of set S then it is not smaller than any other element y : there is no $y \in S$ with $x \preceq y$ and $x \neq y$.

B6.3 Strict Orders

Strict Orders

- ▶ A **partial** order is reflexive, antisymmetric and transitive.
- ▶ We now consider **strict orders**.
- ▶ Example strict order relations are $<$ over \mathbb{N} or \subset for sets.
- ▶ Are these relations
 - ▶ reflexive?
 - ▶ irreflexive?
 - ▶ symmetric?
 - ▶ asymmetric?
 - ▶ antisymmetric?
 - ▶ transitive?

Strict Orders – Definition

Definition (Strict order)

A binary relation \prec over set S is a **strict order** if \prec is **irreflexive, asymmetric and transitive**.

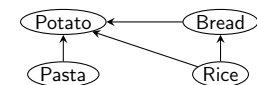
Which of these relations are strict orders?

- ▶ subset relation \subseteq for sets
- ▶ strict superset relation \supset for sets

Can a relation be both, a partial order and a strict order?

Strict Total Orders

- ▶ As partial orders, a strict order does not automatically allow us to rank arbitrary two objects against each other.
- ▶ **Example 1** (personal preferences):
 - ▶ “Pasta tastes better than potato.”
 - ▶ “Rice tastes better than bread.”
 - ▶ “Bread tastes better than potato.”
 - ▶ “Rice tastes better than potato.”
 - ▶ This definition of “tastes better than” is a strict order.
 - ▶ No ranking of pasta against rice or of pasta against bread.
- ▶ **Example 2:** \subset relation for sets
- ▶ It **doesn't work** to simply require that the strict order is total. Why?



Strict Total Orders – Definition

Definition (Trichotomy)

A binary relation R over set S is **trichotomous** if for all $x, y \in S$ exactly one of xRy , yRx or $x = y$ is true.

Definition (Strict total order)

A binary relation \prec over S is a **strict total order** if \prec is **trichotomous** and a **strict order**.

A strict total order completely ranks the elements of set S .

Example: $<$ relation over \mathbb{N}_0 gives the standard ordering $0, 1, 2, 3, \dots$ of natural numbers.

Special Elements

Special elements are defined almost as for partial orders:

Definition (Least/greatest/minimal/maximal element of a set)

Let \prec be a **strict** order over set S .

An element $x \in S$ is the **least element** of S if for all $y \in S$ where $y \neq x$ it holds that $x \prec y$.

It is the **greatest element** of S if for all $y \in S$ where $y \neq x$, $y \prec x$.

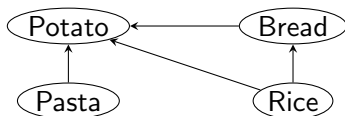
Element $x \in S$ is a **minimal element** of S if there is no $y \in S$ with $y \prec x$.

It is a **maximal element** of S if there is no $y \in S$ with $x \prec y$.

Special Elements – Example

Consider again the previous example:

$S = \{\text{Pasta, Potato, Bread, Rice}\}$
 $\prec = \{(\text{Pasta, Potato}), (\text{Bread, Potato}), (\text{Rice, Potato}), (\text{Rice, Bread})\}$



Is there a least and a greatest element?
 Which elements are maximal or minimal?

Summary and Outlook

- ▶ A **strict order** is **irreflexive**, **asymmetric** and **transitive**.
- ▶ Strict **total** orders and **special elements** are analogously defined as for partial sets but with a special treatment of equal elements.
- ▶ For partial order \preceq we can define a related strict order \prec as $x \prec y$ if $x \preceq y$ and $y \not\preceq x$.
- ▶ For strict order \prec we can define a related partial order \preceq as $x \preceq y$ if $x \prec y$ or $x = y$.
- ▶ There are more related concepts, e. g.
 - ▶ **(total) preorder**: (connex), reflexive, transitive
 - ▶ **well-order**: total order over S such that every non-empty subset has a least element