

Discrete Mathematics in Computer Science

Relations

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- There are also relations of **higher arity**, e. g.
 - “ $x + y = z$ ” for integers x, y, z .
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 - “ $x + y = z$ ” for integers x, y, z .
 - “The name, address and office number belong to the same person.”
- Relations are for example important for relational databases, semantic networks or knowledge representation and reasoning.

Relations

Definition (Relation)

Let S_1, \dots, S_n be sets.

A **relation over S_1, \dots, S_n** is a set $R \subseteq S_1 \times \dots \times S_n$.

The **arity** of R is n .

- A relation of arity n is a set of n -tuples.
- The set contains the tuples for which the informal property is true.

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- $R' = \{(\text{Gabi, Spiegelgasse 1, 04.005}),$
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Properties of Binary Relations

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- Instead of $(x, y) \in R$, we also write xRy , e. g.
 $x \leq y$ instead of $(x, y) \in \leq$
- If the sets are equal, we say “ R is a binary relation over A ” instead of “ R is a binary relation over A and A ”.
- Such a relation over a set is also called a **homogeneous relation** or an **endorelation**.

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How do these properties relate to irreflexivity?

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Special Classes of Relations

- Some important classes of relations are defined in terms of these properties.
 - **Equivalence relation:** reflexive, symmetric, transitive
 - **Partial order:** reflexive, antisymmetric, transitive
 - **Strict order:** irreflexive, asymmetric, transitive
 - ...
- We will consider these and other classes in detail.