

Discrete Mathematics in Computer Science

B5. Relations

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B5.1 Relations

B5.2 Properties of Binary Relations

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Relations: Informally

- ▶ Informally, a relation is some property that is true or false for an (ordered) collection of objects.
- ▶ We already know some relations, e. g.
 - ▶ \subseteq relation for sets
 - ▶ \leq relation for natural numbers
- ▶ These are examples of **binary** relations, considering **pairs of objects**.
- ▶ There are also relations of **higher arity**, e. g.
 - ▶ “ $x + y = z$ ” for integers x, y, z .
 - ▶ “The name, address and office number belong to the same person.”
- ▶ Relations are for example important for relational databases, semantic networks or knowledge representation and reasoning.

Relations

Definition (Relation)

Let S_1, \dots, S_n be sets.

A **relation over S_1, \dots, S_n** is a set $R \subseteq S_1 \times \dots \times S_n$.

The **arity** of R is n .

- ▶ A relation of arity n is a set of n -tuples.
- ▶ The set contains the tuples for which the informal property is true.

Relations: Examples

- ▶ $\subseteq = \{(S, S') \mid S \text{ and } S' \text{ are sets and for every } x \in S \text{ it holds that } x \in S'\}$
- ▶ $\leq = \{(x, y) \mid x, y \in \mathbb{N}_0 \text{ and } x < y \text{ or } x = y\}$
- ▶ $R = \{(x, y, z) \mid x, y, z \in \mathbb{Z} \text{ and } x + y = z\}$
- ▶ $R' = \{(Gabi, Spiegelgasse 1, 04.005), (Salomé, Spiegelgasse 1, 04.002), (Florian, Spiegelgasse 1, 04.005), (Augusto, Spiegelgasse 5, 04.001)\}$

B5.2 Properties of Binary Relations

Binary Relation

A binary relation is a relation of arity 2:

Definition (binary relation)

A **binary relation** is a relation over two sets A and B .

- ▶ Instead of $(x, y) \in R$, we also write xRy , e.g.
 $x \leq y$ instead of $(x, y) \in \leq$
- ▶ If the sets are equal, we say “ R is a binary relation over A ” instead of “ R is a binary relation over A and A ”.
- ▶ Such a relation over a set is also called a **homogeneous relation** or an **endorelation**.

Reflexivity

A **reflexive** relation relates every object to itself.

Definition (reflexive)

A binary relation R over set A is **reflexive**
if for all $a \in A$ it holds that $(a, a) \in R$.

Which of these relations are reflexive?

- ▶ $R = \{(a, a), (a, b), (a, c), (b, a), (b, c), (c, c)\}$ over $\{a, b, c\}$
- ▶ $R = \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, c)\}$ over $\{a, b, c\}$
- ▶ equality relation = on natural numbers
- ▶ less-than relation \leq on natural numbers
- ▶ strictly-less-than relation $<$ on natural numbers

Irreflexivity

A **irreflexive** relation never relates an object to itself.

Definition (irreflexive)

A binary relation R over set A is **irreflexive**
if for all $a \in A$ it holds that $(a, a) \notin R$.

Which of these relations are irreflexive?

- ▶ $R = \{(a, a), (a, b), (a, c), (b, a), (b, c), (c, c)\}$ over $\{a, b, c\}$
- ▶ $R = \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, c)\}$ over $\{a, b, c\}$
- ▶ equality relation = on natural numbers
- ▶ less-than relation \leq on natural numbers
- ▶ strictly-less-than relation $<$ on natural numbers

Symmetry

Definition (symmetric)

A binary relation R over set A is **symmetric**

if for all $a, b \in A$ it holds that $(a, b) \in R$ iff $(b, a) \in R$.

Which of these relations are symmetric?

- ▶ $R = \{(a, a), (a, b), (a, c), (b, a), (c, a), (c, c)\}$ over $\{a, b, c\}$
- ▶ $R = \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, c)\}$ over $\{a, b, c\}$
- ▶ equality relation = on natural numbers
- ▶ less-than relation \leq on natural numbers
- ▶ strictly-less-than relation $<$ on natural numbers

Asymmetry and Antisymmetry

Definition (asymmetric and antisymmetric)

Let R be a binary relation over set A .

Relation R is **asymmetric** if

for all $a, b \in A$ it holds that if $(a, b) \in R$ then $(b, a) \notin R$.

Relation R is **antisymmetric** if for all $a, b \in A$ with $a \neq b$ it holds that if $(a, b) \in R$ then $(b, a) \notin R$.

Which of these relations are asymmetric/antisymmetric?

- ▶ $R = \{(a, a), (a, b), (a, c), (b, a), (c, a), (c, c)\}$ over $\{a, b, c\}$
- ▶ $R = \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, c)\}$ over $\{a, b, c\}$
- ▶ equality relation = on natural numbers
- ▶ less-than relation \leq on natural numbers
- ▶ strictly-less-than relation $<$ on natural numbers

How do these properties relate to irreflexivity?

Transitivity

Definition

A binary relation R over set A is **transitive**
if it holds for all $a, b, c \in A$ that
if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$.

Which of these relations are transitive?

- ▶ $R = \{(a, a), (a, b), (a, c), (b, a), (c, a), (c, c)\}$ over $\{a, b, c\}$
- ▶ $R = \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, c)\}$ over $\{a, b, c\}$
- ▶ equality relation = on natural numbers
- ▶ less-than relation \leq on natural numbers
- ▶ strictly-less-than relation $<$ on natural numbers

Special Classes of Relations

- ▶ Some important classes of relations are defined in terms of these properties.
 - ▶ **Equivalence relation:** reflexive, symmetric, transitive
 - ▶ **Partial order:** reflexive, antisymmetric, transitive
 - ▶ **Strict order:** irreflexive, asymmetric, transitive
 - ▶ ...
- ▶ We will consider these and other classes in detail.