

# Discrete Mathematics in Computer Science

## B5. Relations

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## B5.1 Relations

## B5.2 Properties of Binary Relations

# B5.1 Relations

## Relations: Informally

- ▶ Informally, a relation is some property that is true or false for an (ordered) collection of objects.
- ▶ We already know some relations, e. g.
  - ▶  $\subseteq$  relation for sets
  - ▶  $\leq$  relation for natural numbers
- ▶ These are examples of **binary** relations, considering **pairs of objects**.
- ▶ There are also relations of **higher arity**, e. g.
  - ▶ “ $x + y = z$ ” for integers  $x, y, z$ .
  - ▶ “The name, address and office number belong to the same person.”
- ▶ Relations are for example important for relational databases, semantic networks or knowledge representation and reasoning.

# Relations

## Definition (Relation)

Let  $S_1, \dots, S_n$  be sets.

A **relation over  $S_1, \dots, S_n$**  is a set  $R \subseteq S_1 \times \dots \times S_n$ .

The **arity** of  $R$  is  $n$ .

- ▶ A relation of arity  $n$  is a set of  $n$ -tuples.
- ▶ The set contains the tuples for which the informal property is true.

# Relations: Examples

- ▶  $\subseteq = \{(S, S') \mid S \text{ and } S' \text{ are sets and for every } x \in S \text{ it holds that } x \in S'\}$
- ▶  $\leq = \{(x, y) \mid x, y \in \mathbb{N}_0 \text{ and } x < y \text{ or } x = y\}$
- ▶  $R = \{(x, y, z) \mid x, y, z \in \mathbb{Z} \text{ and } x + y = z\}$
- ▶  $R' = \{(\text{Gabi, Spiegelgasse 1, 04.005}),$   
 $(\text{Salomé, Spiegelgasse 1, 04.002}),$   
 $(\text{Florian, Spiegelgasse 1, 04.005}),$   
 $(\text{Augusto, Spiegelgasse 5, 04.001})\}$

## B5.2 Properties of Binary Relations

# Binary Relation

A binary relation is a relation of arity 2:

## Definition (binary relation)

A **binary relation** is a relation over two sets  $A$  and  $B$ .

- ▶ Instead of  $(x, y) \in R$ , we also write  $xRy$ , e. g.  
 $x \leq y$  instead of  $(x, y) \in \leq$
- ▶ If the sets are equal, we say “ $R$  is a binary relation over  $A$ ” instead of “ $R$  is a binary relation over  $A$  and  $A$ ”.
- ▶ Such a relation over a set is also called a **homogeneous relation** or an **endorelation**.



# Reflexivity

A **reflexive** relation relates every object to itself.

## Definition (reflexive)

A binary relation  $R$  over set  $A$  is **reflexive** if **for all  $a \in A$  it holds that  $(a, a) \in R$ .**

Which of these relations are reflexive?

- ▶  $R = \{(a, a), (a, b), (a, c), (b, a), (b, c), (c, c)\}$  over  $\{a, b, c\}$
- ▶  $R = \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, c)\}$  over  $\{a, b, c\}$
- ▶ equality relation  $=$  on natural numbers
- ▶ less-than relation  $\leq$  on natural numbers
- ▶ strictly-less-than relation  $<$  on natural numbers

# Irreflexivity

A **irreflexive** relation never relates an object to itself.

## Definition (irreflexive)

A binary relation  $R$  over set  $A$  is **irreflexive** if **for all  $a \in A$  it holds that  $(a, a) \notin R$ .**

Which of these relations are irreflexive?

- ▶  $R = \{(a, a), (a, b), (a, c), (b, a), (b, c), (c, c)\}$  over  $\{a, b, c\}$
- ▶  $R = \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, c)\}$  over  $\{a, b, c\}$
- ▶ equality relation  $=$  on natural numbers
- ▶ less-than relation  $\leq$  on natural numbers
- ▶ strictly-less-than relation  $<$  on natural numbers

# Symmetry

## Definition (symmetric)

A binary relation  $R$  over set  $A$  is **symmetric** if for all  $a, b \in A$  it holds that  $(a, b) \in R$  iff  $(b, a) \in R$ .

Which of these relations are symmetric?

- ▶  $R = \{(a, a), (a, b), (a, c), (b, a), (c, a), (c, c)\}$  over  $\{a, b, c\}$
- ▶  $R = \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, c)\}$  over  $\{a, b, c\}$
- ▶ equality relation  $=$  on natural numbers
- ▶ less-than relation  $\leq$  on natural numbers
- ▶ strictly-less-than relation  $<$  on natural numbers

# Asymmetry and Antisymmetry

## Definition (asymmetric and antisymmetric)

Let  $R$  be a binary relation over set  $A$ .

Relation  $R$  is **asymmetric** if

for all  $a, b \in A$  it holds that if  $(a, b) \in R$  then  $(b, a) \notin R$ .

Relation  $R$  is **antisymmetric** if for all  $a, b \in A$  with  $a \neq b$  it holds that if  $(a, b) \in R$  then  $(b, a) \notin R$ .

Which of these relations are asymmetric/antisymmetric?

- ▶  $R = \{(a, a), (a, b), (a, c), (b, a), (c, a), (c, c)\}$  over  $\{a, b, c\}$
- ▶  $R = \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, c)\}$  over  $\{a, b, c\}$
- ▶ equality relation  $=$  on natural numbers
- ▶ less-than relation  $\leq$  on natural numbers
- ▶ strictly-less-than relation  $<$  on natural numbers

How do these properties relate to irreflexivity?

# Transitivity

## Definition

A binary relation  $R$  over set  $A$  is **transitive** if it holds for all  $a, b, c \in A$  that  
**if  $(a, b) \in R$  and  $(b, c) \in R$  then  $(a, c) \in R$ .**

Which of these relations are transitive?

- ▶  $R = \{(a, a), (a, b), (a, c), (b, a), (c, a), (c, c)\}$  over  $\{a, b, c\}$
- ▶  $R = \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, c)\}$  over  $\{a, b, c\}$
- ▶ equality relation  $=$  on natural numbers
- ▶ less-than relation  $\leq$  on natural numbers
- ▶ strictly-less-than relation  $<$  on natural numbers

# Special Classes of Relations

- ▶ Some important classes of relations are defined in terms of these properties.
  - ▶ **Equivalence relation:** reflexive, symmetric, transitive
  - ▶ **Partial order:** reflexive, antisymmetric, transitive
  - ▶ **Strict order:** irreflexive, asymmetric, transitive
  - ▶ ...
- ▶ We will consider these and other classes in detail.