

Discrete Mathematics in Computer Science

Cantor's Theorem

Malte Helmert, Gabriele Röger

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Countable Sets

We already know:

- The **cardinality of \mathbb{N}_0** is \aleph_0 .
- All sets with cardinality \aleph_0 are called **countably infinite**.
- A **countable** set is finite or countably infinite.
- Every subset of a countable set is countable.
- The union of countably many countable sets is countable.

Countable Sets

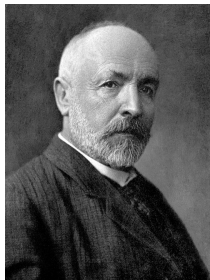
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These questions were still open:

- Do all infinite sets have the same cardinality?
- Does the power set of infinite set S have the same cardinality as S ?

Georg Cantor



- German mathematician (1845–1918)
- Proved that the rational numbers are countable.
- Proved that the real numbers are not countable.
- **Cantor's Theorem:** For every set S it holds that $|S| < |\mathcal{P}(S)|$.

Our Plan

- Understand Cantor's theorem
- Understand an important theoretical implication for computer science

Cantor's Diagonal Argument Illustrated on a Finite Set

$$S = \{a, b, c\}.$$

Consider an arbitrary injective function from S to $\mathcal{P}(S)$.

For example:

	<i>a</i>	<i>b</i>	<i>c</i>	
<i>a</i>	1	0	1	<i>a</i> mapped to $\{a, c\}$
<i>b</i>	1	1	0	<i>b</i> mapped to $\{a, b\}$
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We can identify an “unused” element of $\mathcal{P}(S)$.

Complement the entries on the main diagonal.

Works with every injective function from S to $\mathcal{P}(S)$.

→ there cannot be a bijection from S to $\mathcal{P}(S)$.

Cantor's Diagonal Argument on a Countably Infinite Set

$$S = \mathbb{N}_0.$$

Consider an arbitrary injective function from \mathbb{N}_0 to $\mathcal{P}(\mathbb{N}_0)$.

For example:

	0	1	2	3	4	...
0	1	0	1	0	1	...
1	1	1	0	1	0	...
2	0	1	0	1	0	...
3	1	1	0	0	0	...
4	1	1	0	1	1	...
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\ddots
<hr/>						
	0	0	1	1	0	...

Complementing the entries on the main diagonal again results in an “unused” element of $\mathcal{P}(\mathbb{N}_0)$.

Cantor's Theorem

Theorem (Cantor's Theorem)

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Proof.

We need to show that

- 1 There is an injective function from S to $\mathcal{P}(S)$.
- 2 There is no bijection from S to $\mathcal{P}(S)$.

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For 1, consider function $f : S \rightarrow \mathcal{P}(S)$ with $f(x) = \{x\}$. Each element of S is paired with a unique element of $\mathcal{P}(S)$

Cantor's Theorem

Proof (continued).

For 2, we show for every injective function $f : S \rightarrow \mathcal{P}(S)$ that it is not a bijection from S to $\mathcal{P}(S)$.

This is sufficient because every bijection is injective.

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For 2, we show for every injective function $f : S \rightarrow \mathcal{P}(S)$ that it is not a bijection from S to $\mathcal{P}(S)$.

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Let f be an arbitrary injective function with $f : S \rightarrow \mathcal{P}(S)$.

Consider $M = \{x \mid x \in S, x \notin f(x)\}$.

For every $x \in S$ it holds that $f(x) \neq M$ because $x \in f(x)$ iff not $x \notin f(x)$ iff not $x \in M$ iff $x \notin M$.

Hence, there is no $x \in S$ with $f(x) = M$. As $M \in \mathcal{P}(S)$ this implies that f is not a bijection from S to $\mathcal{P}(S)$. □

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Consequences of Cantor's Theorem

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Infinite Sets can Have Different Cardinalities

There are infinitely many different cardinalities of infinite sets:

- $|\mathbb{N}_0| < |\mathcal{P}(\mathbb{N}_0)| < |\mathcal{P}(\mathcal{P}(\mathbb{N}_0))| < \dots$
- $|\mathbb{N}_0| = \aleph_0 = \beth_0$
- $|\mathcal{P}(\mathbb{N}_0)| = \beth_1 (= |\mathbb{R}|)$
- $|\mathcal{P}(\mathcal{P}(\mathbb{N}_0))| = \beth_2$
- \dots

Existence of Unsolvable Problems

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Why can we say so?

Decision Problems

“Intuitive Definition:” Decision Problem

A **decision problem** is a Yes-No question of the form

“Does the given input have a certain property?”

- “Does the given binary tree have more than three leaves?”
- “Is the given integer odd?”
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- A computer program solves a decision problem if it terminates on every input and returns the correct answer.

More Problems than Programs I

- A computer program is given by a finite string.
- A decision problem corresponds to a set of strings.

More Problems than Programs II

- Consider an arbitrary finite set of symbols (an **alphabet**) Σ .
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 - every subset of S corresponds to a separate decision problem
- By Cantor's theorem $|S| < |\mathcal{P}(S)|$,
so **there are more problems than programs.**

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Sets: Summary

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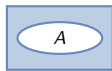
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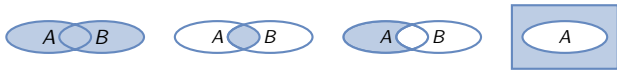
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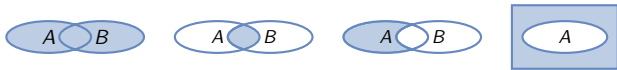
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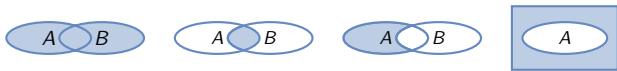
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 - For all sets S it holds that $|S| < |\mathcal{P}(S)|$.