# Discrete Mathematics in Computer Science

## B3. Cantor's Theorem

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## B3.1 Cantor's Theorem

## B3.2 Consequences of Cantor's Theorem

## B3.3 Sets: Summary

# B3.1 Cantor's Theorem

# Countable Sets

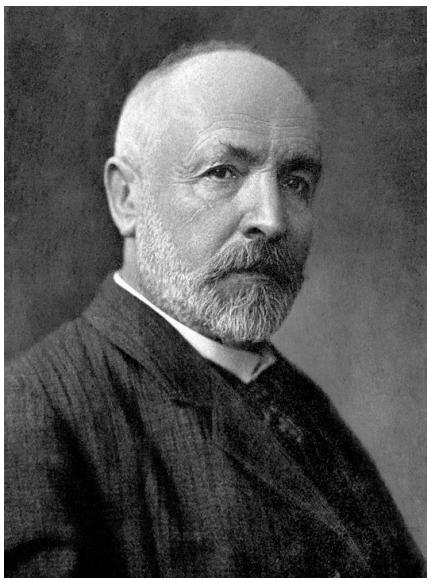
We already know:

- The cardinality of $\mathbb{N}_0$ is $\aleph_0$.
- All sets with cardinality $\aleph_0$ are called countably infinite.
- A countable set is finite or countably infinite.
- Every subset of a countable set is countable.
- The union of countably many countable sets is countable.

These questions were still open:

- Do all infinite sets have the same cardinality?
- Does the power set of infinite set $S$ have the same cardinality as $S$?

# Georg Cantor



- German mathematician (1845–1918)
- Proved that the rational numbers are countable.
- Proved that the real numbers are not countable.
- Cantor's Theorem: For every set $S$ it holds that $|S| < |\mathcal{P}(S)|$.

# Our Plan

- Understand Cantor's theorem
- Understand an important theoretical implication for computer science

# Cantor's Diagonal Argument Illustrated on a Finite Set

$S = \{a, b, c\}$.

Consider an arbitrary injective function from $S$ to $\mathcal{P}(S)$.
For example:

| | $a$ | $b$ | $c$ | |
|---|---|---|---|---|
| $a$ | 1 | 0 | 1 | $a$ mapped to $\{a, c\}$ |
| $b$ | 1 | 1 | 0 | $b$ mapped to $\{a, b\}$ |
| $c$ | 0 | 1 | 0 | $c$ mapped to $\{b\}$ |
| | 0 | 0 | 1 | nothing was mapped to $\{c\}$. |

We can identify an "unused" element of $\mathcal{P}(S)$.
Complement the entries on the main diagonal.

Works with every injective function from $S$ to $\mathcal{P}(S)$.
$\rightarrow$ there cannot be a bijection from $S$ to $\mathcal{P}(S)$.

# Cantor's Diagonal Argument on a Countably Infinite Set

$S = \mathbb{N}_0$.

Consider an arbitrary injective function from $\mathbb{N}_0$ to $\mathcal{P}(\mathbb{N}_0)$.
For example:

| | 0 | 1 | 2 | 3 | 4 | ... |
|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 1 | 0 | 1 | ... |
| 1 | 1 | 1 | 0 | 1 | 0 | ... |
| 2 | 0 | 1 | 0 | 1 | 0 | ... |
| 3 | 1 | 1 | 0 | 0 | 0 | ... |
| 4 | 1 | 1 | 0 | 1 | 1 | ... |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋱ |
| | 0 | 0 | 1 | 1 | 0 | ... |

Complementing the entries on the main diagonal again results in an "unused" element of $\mathcal{P}(\mathbb{N}_0)$.

# Cantor's Theorem

**Theorem (Cantor's Theorem)**

*For every set $S$ it holds that $|S| < |\mathcal{P}(S)|$.*

**Proof.**

We need to show that

1. There is an injective function from $S$ to $\mathcal{P}(S)$.
2. There is no bijection from $S$ to $\mathcal{P}(S)$.

For 1, consider function $f : S \to \mathcal{P}(S)$ with $f(x) = \{x\}$. Each element of $S$ is paired with a unique element of $\mathcal{P}(S)$. ...

# Cantor's Theorem

**Proof (continued).**

For 2, we show for every injective function $f : S \to \mathcal{P}(S)$ that it is not a bijection from $S$ to $\mathcal{P}(S)$.
This is sufficient because every bijection is injective.

Let $f$ be an arbitrary injective function with $f : S \to \mathcal{P}(S)$.

Consider $M = \{x \mid x \in S, x \notin f(x)\}$.

For every $x \in S$ it holds that $f(x) \neq M$ because
$x \in f(x)$ iff not $x \notin f(x)$ iff not $x \in M$ iff $x \notin M$.

Hence, there is no $x \in S$ with $f(x) = M$. As $M \in \mathcal{P}(S)$ this implies that $f$ is not a bijection from $S$ to $\mathcal{P}(S)$. □

# B3.2 Consequences of Cantor's Theorem

# Infinite Sets can Have Different Cardinalities

There are infinitely many different cardinalities of infinite sets:

- $|\mathbb{N}_0| < |\mathcal{P}(\mathbb{N}_0)| < |\mathcal{P}(\mathcal{P}(\mathbb{N}_0))| < \dots$
- $|\mathbb{N}_0| = \aleph_0 = \beth_0$
- $|\mathcal{P}(\mathbb{N}_0)| = \beth_1 (= |\mathbb{R}|)$
- $|\mathcal{P}(\mathcal{P}(\mathbb{N}_0))| = \beth_2$
- $\dots$

# Existence of Unsolvable Problems

There are more problems in computer science
than there are programs to solve them.

There are problems that cannot be solved by a computer program!

Why can we say so?

# Decision Problems

**"Intuitive Definition:" Decision Problem**

A decision problem is a Yes-No question of the form
"Does the given input have a certain property?"

- "Does the given binary tree have more than three leaves?"
- "Is the given integer odd?"
- "Given a train schedule, is there a connection from Basel to Belinzona that takes at most 2.5 hours?"

- Input can be encoded as some finite string.
- Problem can also be represented as the (possibly infinite) set of all input strings where the answer is "yes".
- A computer program solves a decision problem if it terminates on every input and returns the correct answer.

# More Problems than Programs I

- A computer program is given by a finite string.
- A decision problem corresponds to a set of strings.

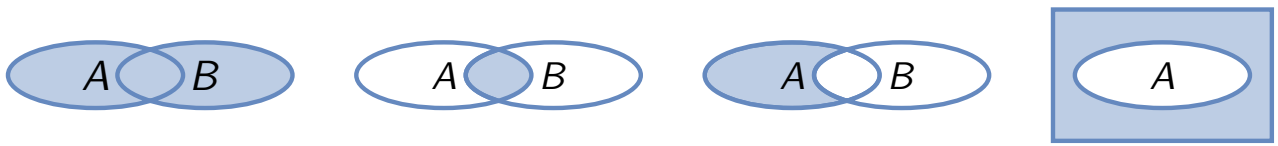
# More Problems than Programs II

- Consider an arbitrary finite set of symbols (an alphabet) $\Sigma$.
- You can think of $\Sigma = \{0, 1\}$
  as internally computers operate on binary representation.
- Let $S$ be the set of all finite strings made from symbols in $\Sigma$.
- There are at most $|S|$ computer programs with this alphabet.
- There are at least $|\mathcal{P}(S)|$ problems with this alphabet.
  - every subset of $S$ corresponds to a separate decision problem
- By Cantor's theorem $|S| < |\mathcal{P}(S)|$,
  so there are more problems than programs.

# B3.3 Sets: Summary

# Summary

- A set is an unordered collection of distinct objects.
- Set operations: union, intersection, set difference, complement




  - Commutativity, associativity and distributivity of union and intersection
  - De Morgan's law: $\overline{A \cup B} = \overline{A} \cap \overline{B}$ and $\overline{A \cap B} = \overline{A} \cup \overline{B}$.
- The cardinality measures the "size" of a set.
  - For finite sets, the cardinality equals the number of elements.
  - All sets with the same cardinality as $\mathbb{N}_0$ are countably infinite.
  - All sets with cardinality $\leq |\mathbb{N}_0|$ are countable.
- The power set $\mathcal{P}(S)$ of set $S$ is the set of all subsets of $S$.
  - For finite sets $S$ it holds that $|\mathcal{P}(S)| = 2^{|S|}$.
  - For all sets $S$ it holds that $|S| < |\mathcal{P}(S)|$.