

Discrete Mathematics in Computer Science

Mathematical Induction

Malte Helmert, Gabriele Röger

University of Basel

Proof Techniques

most common proof techniques:

- direct proof
- indirect proof (proof by contradiction)
- contrapositive
- **mathematical induction**
- structural induction

Mathematical Induction

Concrete Mathematics by Graham, Knuth and Patashnik (p. 3)

Mathematical induction proves that

we can climb as high as we like on a ladder,

by proving that we can climb onto the bottom rung (**the basis**)

and that

from each rung we can climb up to the next one (**the step**).

Propositions

Consider a statement on all natural numbers n with $n \geq m$.

- E.g. “Every natural number $n \geq 2$ can be written as a product of prime numbers.”
 - $P(2)$: “2 can be written as a product of prime numbers.”
 - $P(3)$: “3 can be written as a product of prime numbers.”
 - $P(4)$: “4 can be written as a product of prime numbers.”
 - ...
 - $P(n)$: “ n can be written as a product of prime numbers.”
 - For every natural number $n \geq 2$ proposition $P(n)$ is true.

A **proposition** $P(n)$ is a mathematical statement that is defined in terms of natural number n .

Mathematical Induction

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Proof (of the truth) of proposition $P(n)$
for all natural numbers n with $n \geq m$:

- **basis**: proof of $P(m)$
- **induction hypothesis (IH)**:
suppose that $P(k)$ is true for all k with $m \leq k \leq n$
- **inductive step**: proof of $P(n + 1)$
using the induction hypothesis

Mathematical Induction: Example I

Theorem

For all $n \in \mathbb{N}_0$ with $n \geq 1$: $\sum_{k=1}^n (2k - 1) = n^2$

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Mathematical induction over n :

basis $n = 1$: $\sum_{k=1}^1 (2k - 1) = 2 - 1 = 1 = 1^2$



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inductive step $n \rightarrow n + 1$:

$$\begin{aligned}\sum_{k=1}^{n+1} (2k - 1) &= \left(\sum_{k=1}^n (2k - 1) \right) + 2(n + 1) - 1 \\ &\stackrel{\text{IH}}{=} n^2 + 2(n + 1) - 1 \\ &= n^2 + 2n + 1 = (n + 1)^2\end{aligned}$$



Mathematical Induction: Example II

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Every natural number $n \geq 2$ can be written as a product of prime numbers, i. e. $n = p_1 \cdot p_2 \cdot \dots \cdot p_m$ with prime numbers p_1, \dots, p_m .

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can be written as a product of prime numbers. ...

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- Case 1: $n + 1$ is a prime number \rightsquigarrow trivial



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inductive step $n \rightarrow n + 1$:

- Case 1: $n + 1$ is a prime number \rightsquigarrow trivial
- Case 2: $n + 1$ is not a prime number.

There are natural numbers $2 \leq q, r \leq n$ with $n + 1 = q \cdot r$.

Using IH shows that there are prime numbers

q_1, \dots, q_s with $q = q_1 \cdot \dots \cdot q_s$ and

r_1, \dots, r_t with $r = r_1 \cdot \dots \cdot r_t$.

Together this means $n + 1 = q_1 \cdot \dots \cdot q_s \cdot r_1 \cdot \dots \cdot r_t$.



Weak vs. Strong Induction

- **Weak induction:** Induction hypothesis only supposes that $P(k)$ is true for $k = n$
- **Strong induction:** Induction hypothesis supposes that $P(k)$ is true for all $k \in \mathbb{N}_0$ with $m \leq k \leq n$
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Our previous definition corresponds to **strong induction**.

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Which of the examples had also worked with weak induction?

Is Strong Induction More Powerful than Weak Induction?

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Is Strong Induction More Powerful than Weak Induction?

Are there statements that we can prove with strong induction but not with weak induction?

We can always use a stronger proposition:

- “Every $n \in \mathbb{N}_0$ with $n \geq 2$ can be written as a product of prime numbers.”
- $P(n)$: “ n can be written as a product of prime numbers.”
- $P'(n)$: “all $k \in \mathbb{N}_0$ with $2 \leq k \leq n$ can be written as a product of prime numbers.”

Reformulating Statements

It is sometimes convenient to rephrase a statement.

For example:

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Be careful about how to reformulate a statement!

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Inductively Defined Sets: Examples

Example (Natural Numbers)

The set \mathbb{N}_0 of natural numbers is inductively defined as follows:

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Example (Binary Tree)

The set \mathcal{B} of binary trees is inductively defined as follows:

- \square is a binary tree (a leaf)
- If L and R are binary trees, then $\langle L, \circlearrowright, R \rangle$ is a binary tree (with inner node \circlearrowright).

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- \square is a binary tree (a leaf)
- If L and R are binary trees, then $\langle L, \bigcirc, R \rangle$ is a binary tree (with inner node \bigcirc).

Implicit statement: all elements of the set can be constructed by finite application of these rules

Inductive Definition of a Set

Inductive Definition

A set M can be defined **inductively** by specifying

- **basic elements** that are contained in M
- **construction rules** of the form
“Given some elements of M , another element of M can be constructed like this.”

Structural Induction

Structural Induction

Proof of statement for all elements of an inductively defined set

- **basis**: proof of the statement for the basic elements
- **induction hypothesis (IH)**:
suppose that the statement is true for some elements M
- **inductive step**: proof of the statement for elements constructed by applying a construction rule to M
(one inductive step for each construction rule)

Structural Induction: Example (1)

Definition (Leaves of a Binary Tree)

The number of **leaves** of a binary tree B , written $leaves(B)$, is defined as follows:

$$leaves(\square) = 1$$

$$leaves(\langle L, \circ, R \rangle) = leaves(L) + leaves(R)$$

Definition (Inner Nodes of a Binary Tree)

The number of **inner nodes** of a binary tree B , written $inner(B)$, is defined as follows:

$$inner(\square) = 0$$

$$inner(\langle L, \circ, R \rangle) = inner(L) + inner(R) + 1$$

Structural Induction: Example (2)

Theorem

For all binary trees B : $inner(B) = leaves(B) - 1$.

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Theorem

For all binary trees B : $inner(B) = leaves(B) - 1$.

Proof.

induction basis:

$$inner(\square) = 0 = 1 - 1 = leaves(\square) - 1$$

\rightsquigarrow statement is true for base case

...

Structural Induction: Example (3)

Proof (continued).

induction hypothesis:

to prove that the statement is true for a composite tree $\langle L, \circ, R \rangle$,
we may use that it is true for the subtrees L and R .



Structural Induction: Example (3)

Proof (continued).

induction hypothesis:

to prove that the statement is true for a composite tree $\langle L, \circlearrowleft, R \rangle$, we may use that it is true for the subtrees L and R .

inductive step for $B = \langle L, \circlearrowleft, R \rangle$:

$$\begin{aligned} \mathit{inner}(B) &= \mathit{inner}(L) + \mathit{inner}(R) + 1 \\ &\stackrel{\text{IH}}{=} (\mathit{leaves}(L) - 1) + (\mathit{leaves}(R) - 1) + 1 \\ &= \mathit{leaves}(L) + \mathit{leaves}(R) - 1 = \mathit{leaves}(B) - 1 \end{aligned}$$



Structural Induction: Exercise

Definition (Height of a Binary Tree)

The **height** of a binary tree B , written $height(B)$, is defined as follows:

$$height(\square) = 0$$

$$height(\langle L, \circ, R \rangle) = \max\{height(L), height(R)\} + 1$$

Prove by structural induction:

Theorem

For all binary trees B : $leaves(B) \leq 2^{height(B)}$.