

Discrete Mathematics in Computer Science

What is a Proof?

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What is a **statement**?

Mathematical Statements

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Notes:

- set of preconditions is sometimes empty
- often, “assumptions” is used instead of “preconditions”; slightly unfortunate because “assumption” is also used with another meaning (\rightsquigarrow cf. indirect proofs)

Examples of Mathematical Statements

Examples (some true, some false):

- “Let $p \in \mathbb{N}_0$ be a prime number. Then p is odd.”
- “There exists an even prime number.”
- “Let $p \in \mathbb{N}_0$ with $p \geq 3$ be a prime number. Then p is odd.”
- “All prime numbers $p \geq 3$ are odd.”
- “For all sets A, B, C : $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ ”
- “0 is a natural number.”
- “The equation $a^k + b^k = c^k$ has infinitely many solutions with $a, b, c, k \in \mathbb{N}_1$ and $k \geq 2$.”
- “The equation $a^k + b^k = c^k$ has no solutions with $a, b, c, k \in \mathbb{N}_1$ and $k \geq 3$.”

What are the preconditions, what are the conclusions?

On what Statements can we Build the Proof?

A mathematical proof is

- a sequence of logical steps
- **starting with one set of statements**
- that comes to the conclusion
that some statement must be true.

We can use:

- **axioms**: statements that are assumed to always be true
in the current context
- **theorems** and **lemmas**: statements that were already proven
 - lemma: an intermediate tool
 - theorem: itself a relevant result
- **premises**: assumptions we make
to see what consequences they have

What is a Logical Step?

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Each step directly follows

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- premises,
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- the preconditions of the statement we want to prove.

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For a formal definition, we would need formal logics.

The Role of Definitions

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- A definition introduces an abbreviation.
- Whenever we say “set”, we could instead say “an unordered collection of distinct objects” and vice versa.
- Definitions can also introduce notation.

Disproofs

- A **disproof** (**refutation**) shows that a given mathematical statement is **false** by giving an example where the preconditions are true, but the conclusion is false.
- This requires deriving, in a sequence of proof steps, the opposite (negation) of the conclusion.
- Formally, disproofs are proofs of modified (“negated”) statements.
- Be careful about how to negate a statement!

A Word on Style

A proof should help the reader to see why the result must be true.

- A proof should be easy to follow.
- Omit unnecessary information.
- Move self-contained parts into separate lemmas.
- In complicated proofs, reveal the overall structure in advance.
- Have a clear line of argument.

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→ Writing a proof is like writing an essay.

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Proof Strategies

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Common Forms of Statements

Many statements have one of these forms:

- ① “All $x \in S$ with the property P also have the property Q .”
- ② “ A is a subset of B .”
- ③ “For all $x \in S$: x has property P iff x has property Q .”
- ④ “ $A = B$ ”, where A and B are sets.

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Many statements have one of these forms:

- 1 “All $x \in S$ with the property P also have the property Q .”
- 2 “ A is a subset of B .”
- 3 “For all $x \in S$: x has property P iff x has property Q .”
- 4 “ $A = B$ ”, where A and B are sets.

In the following, we will discuss some typical proof/disproof strategies for such statements.

Proof Strategies

- 1 “All $x \in S$ with the property P also have the property Q .”
“For all $x \in S$: if x has property P , then x has property Q .”
 - To prove, assume you are given an arbitrary $x \in S$ that has the property P .
Give a sequence of proof steps showing that x must have the property Q .
 - To disprove, find a **counterexample**, i. e., find an $x \in S$ that has property P but not Q and prove this.

Proof Strategies

- ② “ A is a subset of B .”
 - To prove, assume you have an arbitrary element $x \in A$ and prove that $x \in B$.
 - To disprove, find an element in $x \in A \setminus B$ and prove that $x \in A \setminus B$.

Proof Strategies

- ③ “For all $x \in S$: x has property P iff x has property Q .”
(“iff”: “if and only if”)
 - To prove, separately prove “if P then Q ” and “if Q then P ”.
 - To disprove, disprove “if P then Q ” or disprove “if Q then P ”.

Proof Strategies

- ④ “ $A = B$ ”, where A and B are sets.
 - To prove, separately prove “ $A \subseteq B$ ” and “ $B \subseteq A$ ”.
 - To disprove, disprove “ $A \subseteq B$ ” or disprove “ $B \subseteq A$ ”.

Proof Techniques

most common proof techniques:

- direct proof
- indirect proof (proof by contradiction)
- contrapositive
- mathematical induction
- structural induction

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Direct Proof

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Direct Proof

Direct Proof

Direct derivation of the statement by deducing or rewriting.

Direct Proof: Example

→ Separate L^AT_EX/PDF file

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Indirect Proof

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Indirect Proof

Indirect Proof (Proof by Contradiction)

- Make an **assumption** that the statement is false.
- Derive a **contradiction** from the assumption together with the preconditions of the statement.
- This shows that the assumption must be false given the preconditions of the statement, and hence the original statement must be true.

Indirect Proof: Example

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Proof by Contrapositive

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Contrapositive

(Proof by) Contrapositive

Prove “If A , then B ” by proving “If not B , then not A .”

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(Proof by) Contrapositive

Prove “If A , then B ” by proving “If not B , then not A .”

Examples:

- Prove “For all $n \in \mathbb{N}_0$: if n^2 is odd, then n is odd” by proving “For all $n \in \mathbb{N}_0$, if n is even, then n^2 is even.”
- Prove “For all $n \in \mathbb{N}_0$: if n is not a square number, then \sqrt{n} is irrational” by proving “For all $n \in \mathbb{N}_0$: if \sqrt{n} is rational, then n is a square number.”

Contrapositive: Example

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Excursus: Computer-assisted Theorem Proving

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Computer-assisted Proofs

- Computers can help proving theorems.
- **Computer-aided proofs** have for example been used for proving theorems by exhaustion.
- Example: **Four color theorem**

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- Example theorem provers: Isabelle/HOL, Lean