Examples for Proof Techniques II

Gabi Röger

1 Proof by Induction

We will consider two ways of proving the following theorem by induction:

Theorem 1. *For every $n \in \mathbb{N}_0$ it holds that if n is odd then $7^n + 3^n$ is divisible by 10.*

Proof. We show this by induction.

We use proposition $P(n)$: "if n is odd then $7^n + 3^n$ is divisible by 10".

Base case $n = 0$: $P(0)$ is trivially true because 0 is not odd.
Base case $n = 1$: $P(1)$ is true because $7^1 + 3^1 = 10$ is divisible by 10.

Induction hypothesis: Suppose $P(k)$ is true for all $0 \leq k \leq n$.

Inductive step: $n \to n + 1$:

If $n + 1$ is even, $P(n + 1)$ is trivially true.

If $n + 1$ is odd, we need to show that $7^{n+1} + 3^{n+1}$ is divisible by 10.

$$\begin{aligned}
7^{n+1} + 3^{n+1} &= 7^2 \cdot 7^{n-1} + 3^2 \cdot 3^{n-1} \\
&= 49 \cdot 7^{n-1} + 9 \cdot 3^{n-1} \\
&= 40 \cdot 7^{n-1} + 9 \cdot 7^{n-1} + 9 \cdot 3^{n-1} \\
&= 40 \cdot 7^{n-1} + 9(7^{n-1} + 3^{n-1}) =: (*)
\end{aligned}$$

Since $n + 1$ is odd, also $n - 1$ must be odd, so by the induction hypothesis we know that there is an integer k such that $10k = 7^{n-1} + 3^{n-1}$. With this k, it holds that

$$\begin{aligned}
(*) = 40 \cdot 7^{n-1} + 9(7^{n-1} + 3^{n-1}) &= 40 \cdot 7^{n-1} + 9 \cdot 10k \\
&= 10 \cdot 4 \cdot 7^{n-1} + 10 \cdot 9k \\
&= 10(4 \cdot 7^{n-1} + 9k).
\end{aligned}$$

As $4 \cdot 7^{n-1} + 9k$ equals an integer, $P(n + 1)$ is also true in the case where $n + 1$ is odd. □

We now rephrase the theorem and again prove this variant by induction:

Theorem 2. *For every $n \in \mathbb{N}_0$ it holds that $7^{2n+1} + 3^{2n+1}$ is divisible by 10.*

Proof. We show this by induction using proposition $P(n)$: "$7^{2n+1} + 3^{2n+1}$ is divisible by 10".

Base case $n = 0$: From $7^{2\cdot 0+1} + 3^{2\cdot 0+1} = 10$ we see that $P(0)$ is true.

Induction hypothesis: Suppose $P(k)$ is true for all $0 \le k \le n$.

Inductive step: $n \to n+1$:

We need to show that $7^{2(n+1)+1} + 3^{2(n+1)+1}$ is divisible by 10.

$$\begin{aligned}
7^{2(n+1)+1} + 3^{2(n+1)+1} &= 7^{2n+3} + 3^{2n+3} \\
&= 7^2 7^{2n+1} + 3^2 3^{2n+1} \\
&= 49 \cdot 7^{2n+1} + 9 \cdot 3^{2n+1} \\
&= 40 \cdot 7^{2n+1} + 9 \cdot 7^{2n+1} + 9 \cdot 3^{2n+1} \\
&= 40 \cdot 7^{2n+1} + 9(7^{2n+1} + 3^{2n+1}) =: (*)
\end{aligned}$$

From the induction hypotheses ($P(n)$ is true), we know that there is an integer k such that $10k = 7^{2n+1} + 3^{2n+1}$. With this k, it holds that

$$\begin{aligned}
(*) = 40 \cdot 7^{2n+1} + 9(7^{2n+1} + 3^{2n+1}) &= 10 \cdot 4 \cdot 7^{2n+1} + 9 \cdot 10k \\
&= 10(4 \cdot 7^{2n+1} + 9k)
\end{aligned}$$

As $(4 \cdot 7^{2n+1} + 9k)$ equals an integer, we can conclude that $7^{2(n+1)+1} + 3^{2(n+1)+1}$ is divisible by 10, so $P(n+1)$ is true. □