Planning and Optimization

F4. Value Iteration

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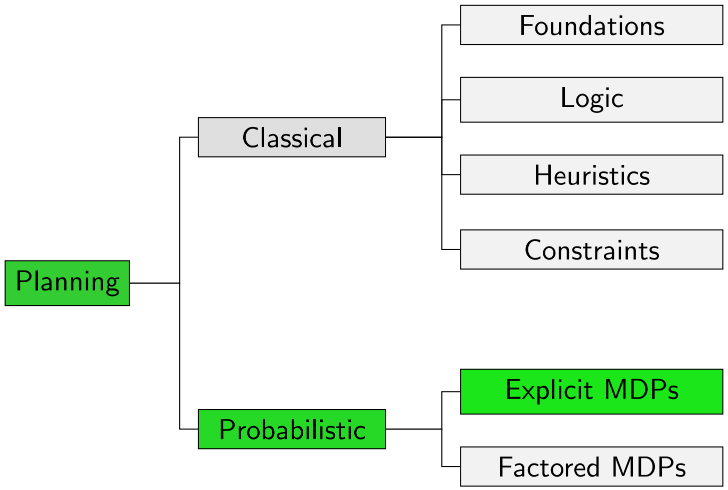
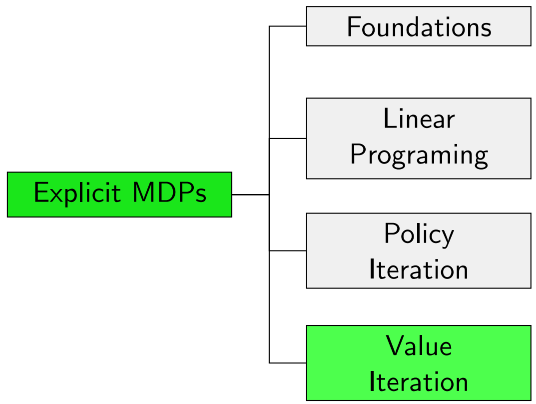


Content of this Course: Explicit MDPs



F4.1 Introduction

From Policy Iteration to Value Iteration

- Policy Iteration:
  - search over policies
  - by evaluating their state-values
- Value Iteration:
  - search directly over state-values
  - optimal policy induced by final state-values

F4.2 Value Iteration

Value Iteration: Idea

- Value Iteration (VI) was first proposed by Bellman in 1957
- computes estimates $\hat{V}^0, \hat{V}^1, \dots$ of $V_\star$ in an iterative process
- starts with arbitrary $\hat{V}^0$
- bases estimate $\hat{V}^{i+1}$ on values of estimate $\hat{V}^i$ by treating Bellman equation as update rule on all states:

$$\hat{V}^{i+1}(s) := \min_{\ell \in L(s)} \left(c(\ell) + \sum_{s' \in S} T(s, \ell, s') \cdot \hat{V}^i(s') \right)$$

  (for SSPs; for MDPs accordingly)
- converges to state-values of optimal policy
- terminates when difference of estimates is small

Example: Value Iteration

	1	2	3	4
5	0.00	0.00	0.00	$s_\star$ 0.00
4	0.00	0.00	0.00	0.00
3	0.00	0.00	0.00	0.00
2	0.00	0.00	0.00	0.00
1	s_0 0.00	0.00	0.00	0.00

$\hat{V}^0$

- cost of 3 to move from striped cells (cost is 1 otherwise)
- moving from gray cells unsuccessful with probability 0.6

Example: Value Iteration

	1	2	3	4
5	1.00	1.00	1.00	$s_\star$ 0.00
4	1.00	1.00	3.00	1.00
3	1.00	1.00	1.00	1.00
2	1.00	1.00	1.00	1.00
1	s_0 1.00	1.00	1.00	1.00

$\hat{V}^1$

- cost of 3 to move from striped cells (cost is 1 otherwise)
- moving from gray cells unsuccessful with probability 0.6

Example: Value Iteration

	1	2	3	4
5	2.00	2.00	1.00	$s_\star$ 0.00
4	2.00	2.00	5.20	1.60
3	2.00	2.00	2.00	2.00
2	2.00	2.00	2.00	2.00
1	s_0 2.00	2.00	2.00	2.00

$\hat{V}^2$

- cost of 3 to move from striped cells (cost is 1 otherwise)
- moving from gray cells unsuccessful with probability 0.6

Example: Value Iteration

	1	2	3	4
5	3.96	2.00	1.00	$s_\star$ 0.00
4	4.60	3.00	7.79	2.31
3	5.00	4.00	4.49	3.96
2	5.00	5.00	4.84	4.76
1	s_0 5.00	5.00	5.00	4.97

$\hat{V}^5$

- cost of 3 to move from striped cells (cost is 1 otherwise)
- moving from gray cells unsuccessful with probability 0.6

Example: Value Iteration

	1	2	3	4
5	4.46	2.00	1.00	$s_\star$ 0.00
4	5.43	3.00	8.44	2.48
3	6.38	4.00	5.00	4.87
2	8.30	6.38	6.00	6.95
1	s_0 8.18	7.31	7.00	8.50

$\hat{V}^{10}$

- cost of 3 to move from striped cells (cost is 1 otherwise)
- moving from gray cells unsuccessful with probability 0.6

Example: Value Iteration

	1	2	3	4
5	4.50	2.00	1.00	$s_\star$ 0.00
4	5.50	3.00	8.50	2.50
3	6.50	4.00	5.00	5.00
2	8.99	6.50	6.00	7.49
1	s_0 8.50	7.50	7.00	9.49

$\hat{V}^{20}$

- cost of 3 to move from striped cells (cost is 1 otherwise)
- moving from gray cells unsuccessful with probability 0.6

Example: Value Iteration

	1	2	3	4
5	4.50	2.00	1.00	$s_\star$ 0.00
4	5.50	3.00	8.50	2.50
3	6.50	4.00	5.00	5.00
2	9.00	6.50	6.00	7.50
1	s_0 8.50	7.50	7.00	9.50

$\hat{V}^{29}$

- cost of 3 to move from striped cells (cost is 1 otherwise)
- moving from gray cells unsuccessful with probability 0.6

Example: Value Iteration

	1	2	3	4
5	$\Rightarrow$ 4.50	$\Rightarrow$ 2.00	$\Rightarrow$ 1.00	$s_\star$ 0.00
4	$\Rightarrow$ 5.50	$\Uparrow$ 3.00	$\Uparrow$ 8.50	$\Uparrow$ 2.50
3	$\Rightarrow$ 6.50	$\Uparrow$ 4.00	$\Leftarrow$ 5.00	$\Uparrow$ 5.00
2	$\Uparrow$ 9.00	$\Uparrow$ 6.50	$\Uparrow$ 6.00	$\Uparrow$ 7.50
1	$\Rightarrow$ s_0 8.50	$\Uparrow$ 7.50	$\Uparrow$ 7.00	$\Leftarrow$ 9.50

$\pi_\star$

- cost of 3 to move from striped cells (cost is 1 otherwise)
- moving from gray cells unsuccessful with probability 0.6

Value Iteration for SSPs

Value Iteration for SSP $\mathcal{T}$ and $\epsilon > 0$

initialize $\hat{V}^0$ arbitarily
for $i = 1, 2, \ldots$:
  for all states $s \in S$:
    $\hat{V}^{i+1}(s) := \min_{\ell \in L(s)} \left(c(\ell) + \sum_{s' \in S} T(s, \ell, s') \cdot \hat{V}^i(s') \right)$
  if $\max_{s \in S} |\hat{V}^{i+1}(s) - \hat{V}^i(s)| < \epsilon$:
    return $\pi_{\hat{V}^{i+1}}$

Value Iteration for MDPs

Value Iteration for MDP $\mathcal{T}$ and $\epsilon > 0$

initialize $\hat{V}^0$ arbitarily
for $i = 1, 2, \ldots$:
  for all states $s \in S$:
    $\hat{V}^{i+1}(s) := \max_{\ell \in L(s)} (R(s) + \gamma \cdot \sum_{s' \in S} T(s, \ell, s') \cdot \hat{V}^i(s'))$
  if $\max_{s \in S} |\hat{V}^{i+1}(s) - \hat{V}^i(s)| < \epsilon$:
    return $\pi_{\hat{V}^{i+1}}$

F4.3 Asynchronous VI

Asynchronous Value Iteration

- Updating all states simultaneously is called synchronous backup
- Asynchronous VI performs backups for individual states
- Different approaches lead to different backup orders
- Can significantly reduce computation
- Guaranteed to converge if all states are selected repeatedly

$\Rightarrow$ Optimal VI with asynchronous backups possible

Example: Asynchronous Value Iteration

	1	2	3	4
5	4.50	2.00	1.00	$s_\star$ 0.00
4	5.50	3.00	8.50	2.50
3	6.50	4.00	5.00	5.00
2	9.00	6.50	6.00	7.50
1	s_0 8.50	7.50	7.00	9.50

$\hat{V}^{57}$

Demo: Asynchronous VI variant that performs backup on each state with probability 0.5

In-place Value Iteration

- Synchronous value iteration creates new copy of value function (two are required simultaneously)

$$\hat{V}^{i+1}(s) := \min_{\ell \in L(s)} \left(c(\ell) + \sum_{s' \in S} T(s, \ell, s') \cdot \hat{V}^{i}(s') \right)$$

- In-place value iteration only requires a single copy of value function

$$\hat{V}(s) := \min_{\ell \in L(s)} \left(c(\ell) + \sum_{s' \in S} T(s, \ell, s') \cdot \hat{V}(s') \right)$$

- In-place VI is asynchronous because some backups are based on "old" values, some on "new" values

Example: In-place Value Iteration

	1	2	3	4
5	4.50	2.00	1.00	$s_\star$ 0.00
4	5.50	3.00	8.50	2.50
3	6.50	4.00	5.00	5.00
2	9.00	6.50	6.00	7.50
1	s_0 8.50	7.50	7.00	9.50

$\hat{V}^{18}$

Demo: Result for in-place value iteration

F4.4 Summary

Linear Programming, Policy Iteration, or Value Iteration?

- Linear Programming is the only technique where the solution is guaranteed to be optimal (independent from ϵ)
- PI and VI are often faster than LP
- PI faster than VI if few iterations required
- VI faster than PI if number of PI iterations outweighs difference of policy evaluation compared to VI
- Asynchronous VI is basis of more sophisticated algorithm that can be applied in large MDPs and SSPs

Summary

- Value Iteration searches in the space of state-values
- VI applies Bellman equation as update rule iteratively
- VI converges to optimal state-values
- VI remains optimal with asynchronous backups as long as all states are selected repeatedly