Planning and Optimization

F3. Blind Methods: Policy Iteration

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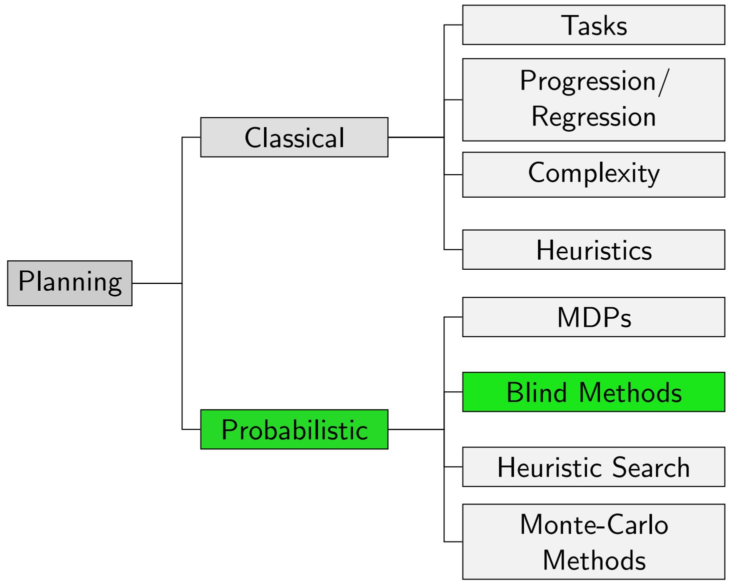
Content of this Course



F3.1 Policy Evaluation

Expected Values under Uncertainty

Definition (Expected Value of a Random Variable)

Let V be a random variable with $n \in \mathbb{N}$ outcomes $d_1, \dots, d_n \in \mathbb{R}$, and let d_i for $i = 1, \dots, n$ occur with probability $p_i \in [0, 1]$ s.t. $\sum_{i=1}^n p_i = 1$.

The expected value of X is $\mathbb{E}[X] = \sum_{i=1}^n (p_i \cdot d_i)$.

Example: Expected Values under Uncertainty

Example

The expected payoff of placing one bet in Swiss Lotto for a cost of 2.50 with (simplified) payout structure

- $d_1 = 30.000.000$ with $p_1 = \frac{1}{31474716}$ (6+1)
- $d_2 = 1.000.000$ with $p_2 = \frac{1}{5245786}$ (6)
- $d_4 = 5.000$ with $p_4 = \frac{1}{850668}$ (5)
- $d_4 = 50$ with $p_4 = \frac{1}{111930}$ (4)
- $d_5 = 10$ with $p_5 = \frac{1}{11480}$ is (3)

$$\mathbb{E}[X] = (\frac{30000000}{31474716} + \frac{1000000}{5245786} + \frac{5000}{850668} + \frac{50}{111930} + \frac{10}{11480}) - 2.5 \approx -1.35.$$

Proper SSP Policy

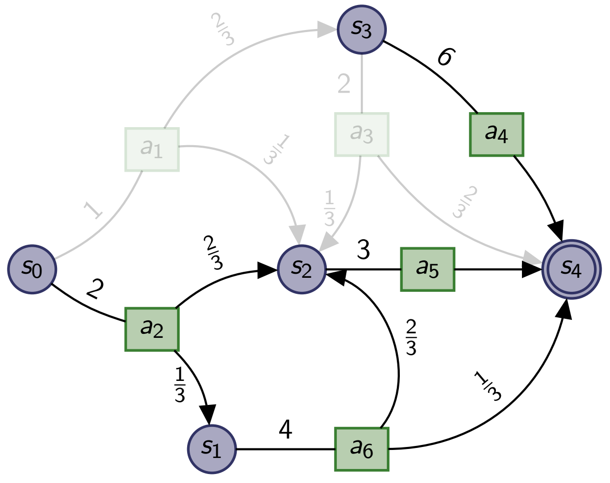
Definition (Proper SSP Policy)

Let $\mathcal{T} = \langle S, L, c, T, s_0, S_\star \rangle$ be an SSP and π be a policy for $\mathcal{T}$. π is proper if it reaches a goal state from each state with probability 1, i.e. if

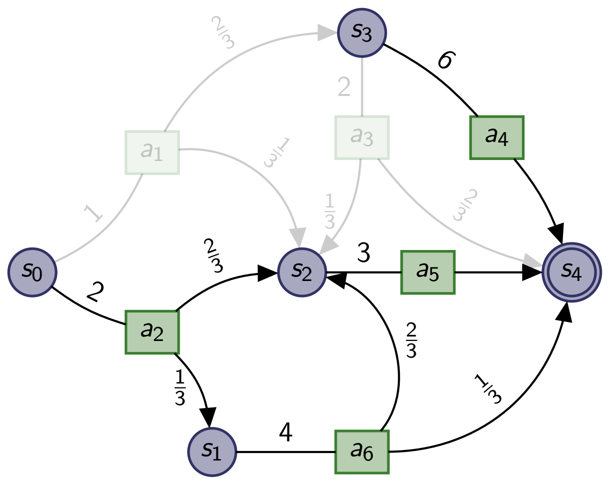
$$\sum_{s \xrightarrow{p_1:\ell_1} s', \dots, s'' \xrightarrow{p_n:\ell_n} s_\star} \prod_{i=1}^{n} p_i = 1$$

for all states $s \in S$.

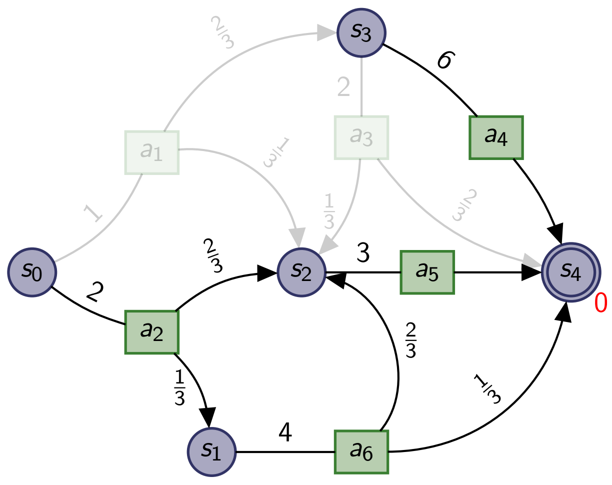
Example: Policy Evaluation for Proper SSP Policy



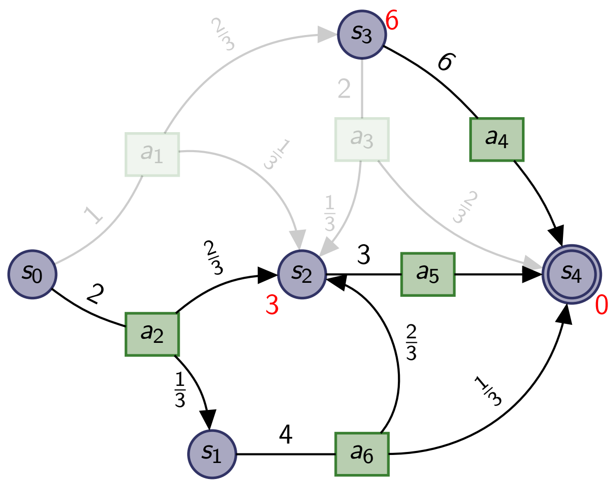
Example: Policy Evaluation for Acyclic Proper SSP Policy



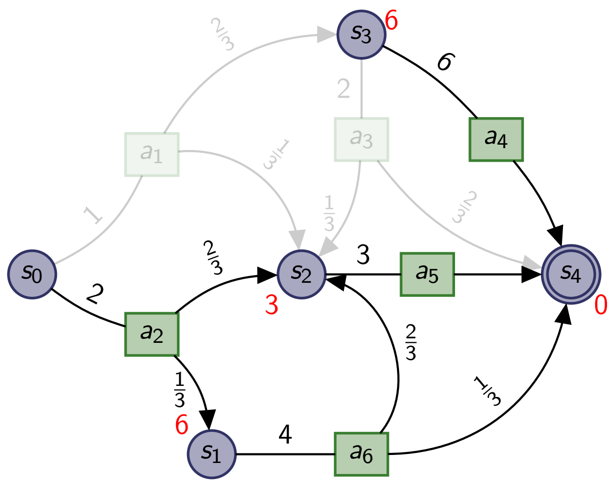
Example: Policy Evaluation for Acyclic Proper SSP Policy



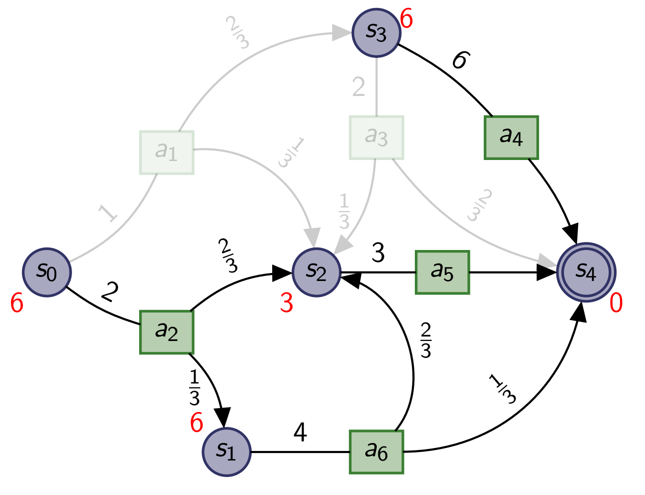
Example: Policy Evaluation for Acyclic Proper SSP Policy



Example: Policy Evaluation for Acyclic Proper SSP Policy



Example: Policy Evaluation for Acyclic Proper SSP Policy



Policy Evaluation for Acyclic Proper SSP Policy

Acyclic Policy Evaluation for SSP $\mathcal{T}$ and complete policy π

```
initialize $V_\pi(s) := \bot$ for all $s \in S$
while there is a $s \in S$ with $V_\pi(s) = \bot$:
    pick $s \in S$ with $V_\pi(s) = \bot$ and
        $V_\pi(s') \neq \bot$ for all $s' \in \text{succ}(s, \pi(s))$
    set $V_\pi(s) := c(\pi(s)) + \sum_{s' \in \text{succ}(s,\pi(s))} T(s, \pi(s), s') \cdot V_\pi(s')$
return $V_\pi$
```

Note: can be generalized to executable policies

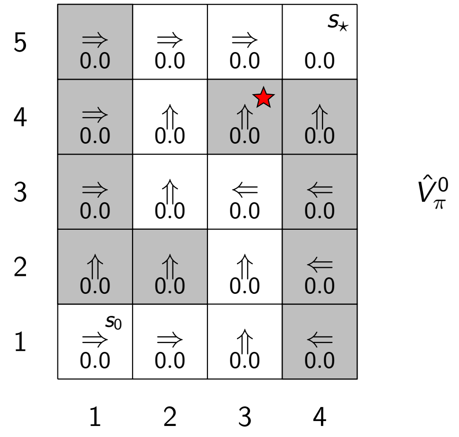
Iterative Policy Evaluation for SSPs

- impossible to compute state-values in one sweep over the state space in presence of cycles
- iterative refinement of $\hat{V}^{i-1}$ to $\hat{V}^i$ possible:

$$\hat{V}^i_\pi(s) = c(\pi(s)) + \sum_{s' \in \text{succ}(s,\pi(s))} T(s, \pi(s), s') \cdot \hat{V}^{i-1}_\pi(s')$$

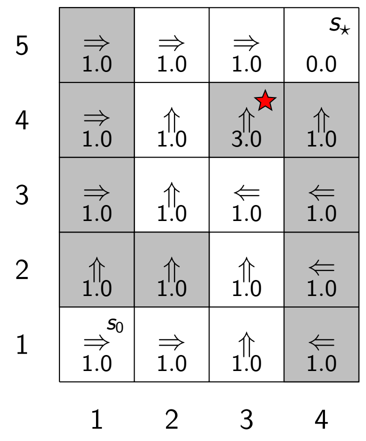
- iterative policy evaluation converges to the true state-values of proper π, i.e., $\lim_{i \to \infty} \hat{V}^i_\pi = V_\pi$
- converges regardless of $\hat{V}^0_\pi$

Example: Iterative Policy Evaluation for SSPs



- cost of 1 for all actions except for moving away from (3,4) where cost is 3
- get stuck when moving away from gray cells with prob. 0.6

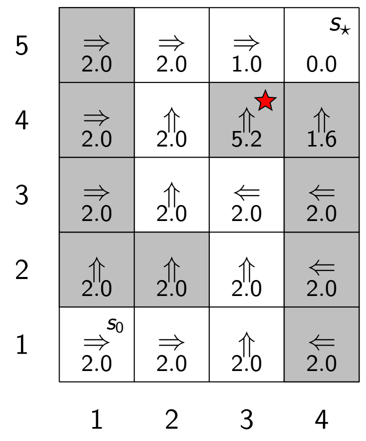
Example: Iterative Policy Evaluation for SSPs



$\hat{V}_\pi^1$

- cost of 1 for all actions except for moving away from (3,4) where cost is 3
- get stuck when moving away from gray cells with prob. 0.6

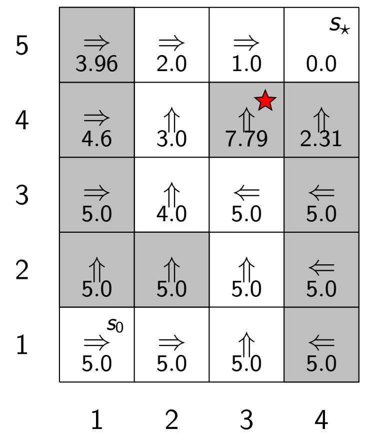
Example: Iterative Policy Evaluation for SSPs



$\hat{V}_\pi^2$

- cost of 1 for all actions except for moving away from (3,4) where cost is 3
- get stuck when moving away from gray cells with prob. 0.6

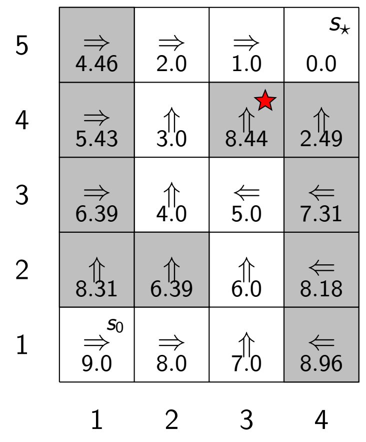
Example: Iterative Policy Evaluation for SSPs



$\hat{V}_\pi^5$

- cost of 1 for all actions except for moving away from (3,4) where cost is 3
- get stuck when moving away from gray cells with prob. 0.6

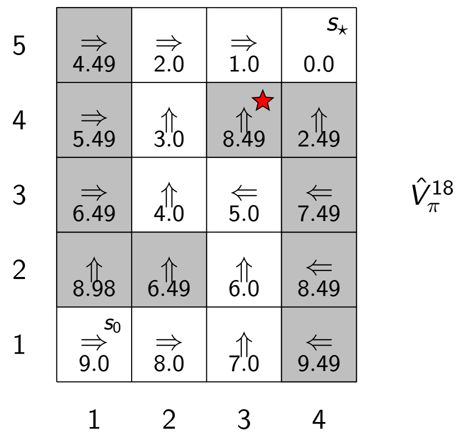
Example: Iterative Policy Evaluation for SSPs



$\hat{V}_\pi^{10}$

- cost of 1 for all actions except for moving away from (3,4) where cost is 3
- get stuck when moving away from gray cells with prob. 0.6

Example: Iterative Policy Evaluation for SSPs



- cost of 1 for all actions except for moving away from (3,4) where cost is 3
- get stuck when moving away from gray cells with prob. 0.6

Iterative Policy Evaluation

Iterative Policy Evaluation for SSP $\mathcal{T}$, policy π and $\epsilon > 0$

```
initialize $\hat{V}^0$ arbitarily
for $i = 1, 2, \dots$:
    for all states $s \in S$:
        $\hat{V}^i_\pi(s) := c(\pi(s)) + \sum_{s' \in S} T(s, \pi(s), s') \cdot \hat{V}^{i-1}_\pi(s')$
    if $\max_{s \in S} |\hat{V}^i_\pi(s) - \hat{V}^{i-1}_\pi(s)| < \epsilon$:
        return $\hat{V}^i_\pi$
```

Note: can be generalized to executable policies

Policy Evaluation: DR-MDPs

What about policy evaluation for DR-MDPs?

- DR-MDPs (with finite state set) are always cyclic
  $\Rightarrow$ acyclic policy evaluation not applicable
- But: existence of goal state not required for iterative policy evaluation
- albeit traces are infinite, iterative policy evaluation converges due to discount factor in DR-MDPs

$\Rightarrow$ use iterative policy evaluation

Policy Evaluation: FH-MDPs

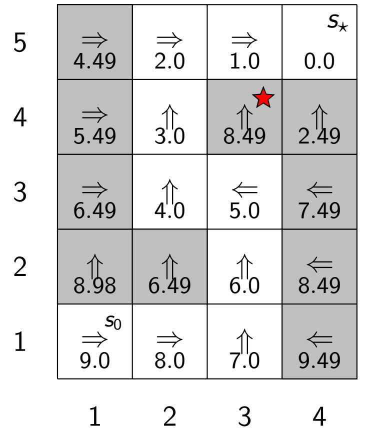
What about policy evaluation for FH-MDPs?

- The relevant state space for FH-MDPs consists of pairs of states and steps-to-go
- as each transition includes a decrease of the steps-to-go, the state space is always acyclic

$\Rightarrow$ use acyclic policy evaluation

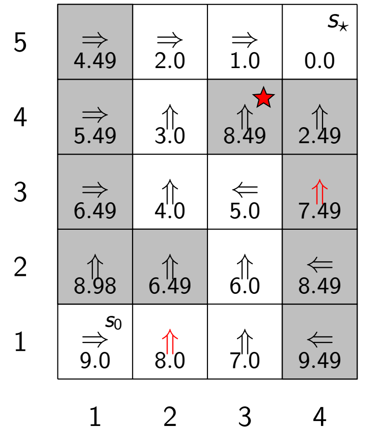
F3.2 Policy Iteration

Example: Greedy Action



- Can we learn more from this than the state-values of a policy?

Example: Greedy Action



- Can we learn more from this than the state-values of a policy?
- Yes! By evaluating all state-action pairs we can derive a better policy

Greedy actions and policies

Definition (Greedy Action)

Let s be a state of an SSP or DR-MDP $\mathcal{T}$ and V be a state-value function for $\mathcal{T}$. The greedy action in s with respect to V is

$$a_V(s) := \arg\min_{\ell \in L(s)} c(\ell) + \sum_{s' \in \mathrm{succ}(s,\ell)} T(s,\ell,s') \cdot V(s').$$

The greedy policy is the policy π_V with $\pi_V(s) = a_V(s)$.

Note: V is often derived as $V_{\pi'}$ from a policy π', but we allow for arbitrary state-value functions that map each state to a real value.

Policy Iteration

- Policy Iteration (PI) was first proposed by Howard in 1960
- exploits observation that greedy actions in result of policy evaluation describe better policy
- starts with arbitrary policy π_0
- alternates policy evaluation and policy improvement
- until convergence to an optimal policy (when policy doesn't change between two steps)

Example: Policy Iteration

5	⇒ 4.49	⇒ 2.0	⇒ 1.0	$s_\star$ 0.0
4	⇒ 5.49	⇑ 3.0	★ ⇑ 8.49	⇑ 2.49
3	⇒ 6.49	⇑ 4.0	⇐ 5.0	⇐ 7.49
2	⇑ 8.98	⇑ 6.49	⇑ 6.0	⇐ 8.49
1	s_0 ⇒ 9.0	⇒ 8.0	⇑ 7.0	⇐ 9.49
	1	2	3	4

π_0

Example: Policy Iteration

5	⇒ 4.49	⇒ 2.0	⇒ 1.0	$s_\star$ 0.0
4	⇒ 5.49	⇑ 3.0	★ ⇑ 8.49	⇑ 2.49
3	⇒ 6.49	⇑ 4.0	⇐ 5.0	⇑ 4.98
2	⇑ 8.98	⇑ 6.49	⇑ 6.0	⇐ 8.49
1	s_0 ⇒ 8.49	⇑ 7.49	⇑ 7.0	⇐ 9.49
	1	2	3	4

π_1

Example: Policy Iteration

5	⇒ 4.49	⇒ 2.0	⇒ 1.0	$s_\star$ 0.0
4	⇒ 5.49	⇑ 3.0	★ ⇑ 8.49	⇑ 2.49
3	⇒ 6.49	⇑ 4.0	⇐ 5.0	⇑ 4.98
2	⇑ 8.98	⇑ 6.49	⇑ 6.0	⇑ 7.47
1	s_0 ⇒ 8.49	⇑ 7.49	⇑ 7.0	⇐ 9.49
	1	2	3	4

$\pi_2 = \pi_3$

Policy Iteration

Policy Iteration for SSP, FH-MDP or DR-MDP $\mathcal{T}$

```
initialize $\pi_0$ to any policy (for SSP: proper)
for $i = 1, 2, \dots$:
    compute $V_{\pi_i}$
    let $\pi_{i+1}$ be the greedy policy w.r.t $V_{\pi_i}$
    if $\pi_i = \pi_{i+1}$:
        return $\pi_i$
```

F3.3 Summary

Summary

- Policy evaluation for acyclic policy is possible in one sweep over the state space.
- Iterative policy evaluation converges over multiple sweeps to true state-values.
- Greedy actions in evaluated policy allow to improve policy.
- Policy iteration alternates policy evaluation and policy improvement.
- Policy iteration results in optimal policy.