Planning and Optimization

D1. Abstractions: Formal Definition and Heuristics

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October 29, 2018

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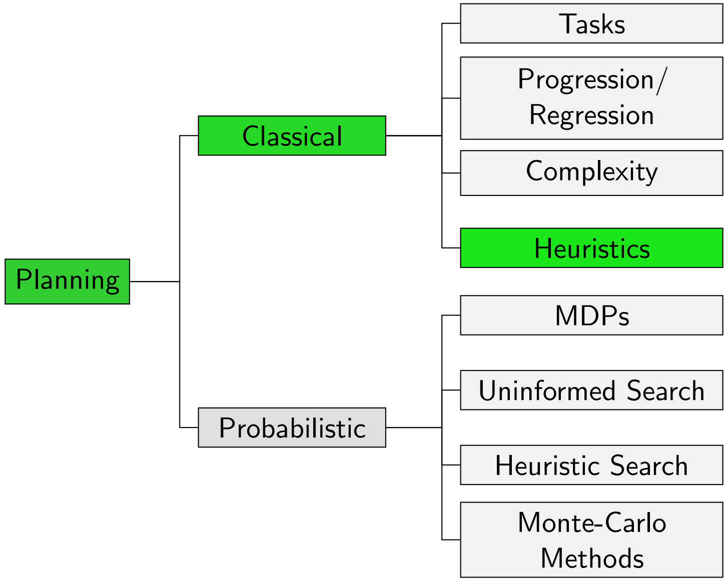
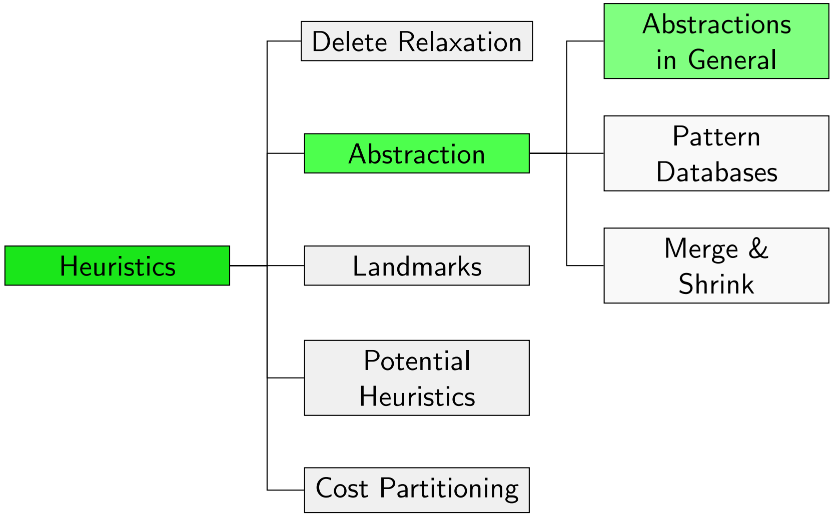


Content of this Course: Heuristics



D1.1 Abstractions

Abstracting a Transition System

Abstracting a transition system means dropping some distinctions between states, while preserving the transition behaviour as much as possible.

- An abstraction of a transition system $\mathcal{T}$ is defined by an abstraction mapping α that defines which states of $\mathcal{T}$ should be distinguished and which ones should not.
- From $\mathcal{T}$ and α, we compute an abstract transition system $\mathcal{T}^\alpha$ which is similar to $\mathcal{T}$, but smaller.
- The abstract goal distances (goal distances in $\mathcal{T}^\alpha$) are used as heuristic estimates for goal distances in $\mathcal{T}$.

Computing the Abstract Transition System

Given $\mathcal{T}$ and α, how do we compute $\mathcal{T}^\alpha$?

Requirement

We want to obtain an admissible heuristic.
Hence, $h^*(\alpha(s))$ (in the abstract state space $\mathcal{T}^\alpha$) should never overestimate $h^*(s)$ (in the concrete state space $\mathcal{T}$).

An easy way to achieve this is to ensure that all solutions in $\mathcal{T}$ are also present in $\mathcal{T}^\alpha$:

- If s is a goal state in $\mathcal{T}$, then $\alpha(s)$ is a goal state in $\mathcal{T}^\alpha$.
- If $\mathcal{T}$ has a transition from s to t, then $\mathcal{T}^\alpha$ has a transition from $\alpha(s)$ to $\alpha(t)$.

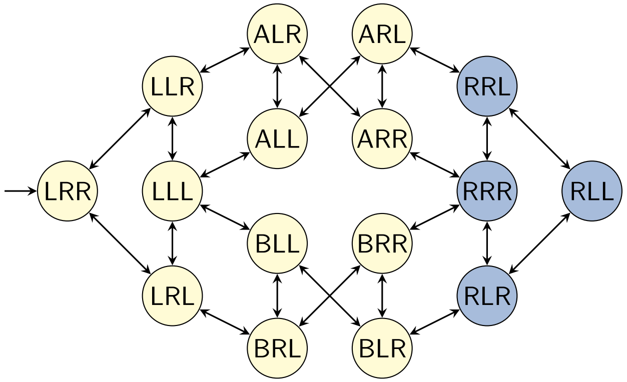
Example Task: One Package, Two Trucks

Example (One Package, Two Trucks)

Consider the following FDR planning task $\langle V, I, O, \gamma\rangle$:

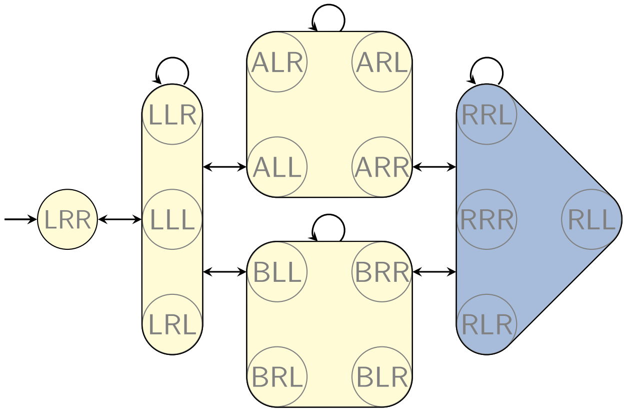
- $V = \{p, t_\mathsf{A}, t_\mathsf{B}\}$ with
  - $\mathrm{dom}(p) = \{\mathsf{L}, \mathsf{R}, \mathsf{A}, \mathsf{B}\}$
  - $\mathrm{dom}(t_\mathsf{A}) = \mathrm{dom}(t_\mathsf{B}) = \{\mathsf{L}, \mathsf{R}\}$
- $I = \{p \mapsto \mathsf{L}, t_\mathsf{A} \mapsto \mathsf{R}, t_\mathsf{B} \mapsto \mathsf{R}\}$
- $O = \{\mathrm{pickup}_{i,j} \mid i \in \{\mathsf{A}, \mathsf{B}\}, j \in \{\mathsf{L}, \mathsf{R}\}\}$
  $\cup \{\mathrm{drop}_{i,j} \mid i \in \{\mathsf{A}, \mathsf{B}\}, j \in \{\mathsf{L}, \mathsf{R}\}\}$
  $\cup \{\mathrm{move}_{i,j,j'} \mid i \in \{\mathsf{A}, \mathsf{B}\}, j, j' \in \{\mathsf{L}, \mathsf{R}\}, j \neq j'\}$, where
  - $\mathrm{pickup}_{i,j} = \langle t_i = j \land p = j, p := i, 1\rangle$
  - $\mathrm{drop}_{i,j} = \langle t_i = j \land p = i, p := j, 1\rangle$
  - $\mathrm{move}_{i,j,j'} = \langle t_i = j, t_i := j', 1\rangle$
- $\gamma = (p = \mathsf{R})$

Concrete Transition System of Example Task



- State $\{p \mapsto i, t_\mathsf{A} \mapsto j, t_\mathsf{B} \mapsto k\}$ is depicted as ijk.
- Transition labels are again not shown. For example, the transition from LLL to ALL has the label pickup$_{\mathsf{A},\mathsf{L}}$.

Abstract Transition System of Example Task



- State $\{p \mapsto i, t_\mathsf{A} \mapsto j, t_\mathsf{B} \mapsto k\}$ is depicted as ijk.
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Abstractions

Definition (Abstraction)

Let $\mathcal{T} = \langle S, L, c, T, s_0, S_\star \rangle$ be a transition system.

An abstraction (also: abstraction function, abstraction mapping) of $\mathcal{T}$ is a function $\alpha : S \to S^\alpha$ defined on the states of $\mathcal{T}$, where S^α is an arbitrary set.

Without loss of generality, we require that α is surjective.

Intuition: α maps the states of $\mathcal{T}$ to another (usually smaller) abstract state space.

Abstract Transition System

Definition (Abstract Transition System)

Let $\mathcal{T} = \langle S, L, c, T, s_0, S_\star \rangle$ be a transition system, and let $\alpha : S \to S^\alpha$ be an abstraction of $\mathcal{T}$.

The abstract transition system induced by α, in symbols $\mathcal{T}^\alpha$, is the transition system $\mathcal{T}^\alpha = \langle S^\alpha, L, c, T^\alpha, s_0^\alpha, S_\star^\alpha \rangle$ defined by:

- $T^\alpha = \{\langle \alpha(s), \ell, \alpha(t) \rangle \mid \langle s, \ell, t \rangle \in T\}$
- $s_0^\alpha = \alpha(s_0)$
- $S_\star^\alpha = \{\alpha(s) \mid s \in S_\star\}$

Terminology

Let $\mathcal{T}$ be a transition system and α be an abstraction of $\mathcal{T}$.

- $\mathcal{T}$ is called the concrete transition system.
- $\mathcal{T}^\alpha$ is called the abstract transition system.
- Similarly: concrete/abstract state space, concrete/abstract transition, etc.

Practical Requirements for Abstractions

To be useful in practice, an abstraction heuristic must be efficiently computable. This gives us two requirements for α:

- For a given state s, the abstract state $\alpha(s)$ must be efficiently computable.
- For a given abstract state $\alpha(s)$, the abstract goal distance $h^*(\alpha(s))$ must be efficiently computable.

There are a number of ways of achieving these requirements:

- pattern database heuristics (Culberson & Schaeffer, 1996)
- merge-and-shrink abstractions (Dräger, Finkbeiner & Podelski, 2006)
- Cartesian abstractions (Ball, Podelski & Rajamani, 2001)
- structural patterns (Katz & Domshlak, 2008b)

D1.2 Homomorphisms and Isomorphisms

Homomorphisms and Isomorphisms

- The abstraction mapping α that transforms $\mathcal{T}$ to $\mathcal{T}^\alpha$ is also called a strict homomorphism from $\mathcal{T}$ to $\mathcal{T}^\alpha$.
- Roughly speaking, in mathematics a homomorphism is a property-preserving mapping between structures.
- A strict homomorphism is one where no additional features are introduced. A non-strict homomorphism in planning would mean that the abstract transition system may include additional transitions and goal states not induced by α.
- We only consider strict homomorphisms in this course.
- If α is bijective, it is called an isomorphism between $\mathcal{T}$ and $\mathcal{T}^\alpha$, and the two transition systems are called isomorphic.

Isomorphic Transition Systems

The notion of isomorphic transition systems is important enough to warrant a formal definition:

Definition (Isomorphic Transition Systems)

Let $\mathcal{T} = \langle S, L, c, T, s_0, S_\star \rangle$ and $\mathcal{T}' = \langle S', L', c', T', s'_0, S'_\star \rangle$ be transition systems.

We say that $\mathcal{T}$ is isomorphic to $\mathcal{T}'$, in symbols $\mathcal{T} \sim \mathcal{T}'$, if there exist bijective functions $\varphi : S \to S'$ and $\lambda : L \to L'$ such that:

- $s \xrightarrow{\ell} t \in T$ iff $\varphi(s) \xrightarrow{\lambda(\ell)} \varphi(t) \in T'$,
- $c'(\lambda(\ell)) = c(\ell)$ for all $\ell \in L$,
- $\varphi(s_0) = s'_0$, and
- $s \in S_\star$ iff $\varphi(s) \in S'_\star$.

Graph-Equivalent Transition Systems

Sometimes a weaker notion of equivalence is useful:

Definition (Graph-Equivalent Transition Systems)

Let $\mathcal{T} = \langle S, L, c, T, s_0, S_\star \rangle$ and $\mathcal{T}' = \langle S', L', c, T', s'_0, S'_\star \rangle$ be transition systems.

We say that $\mathcal{T}$ is graph-equivalent to $\mathcal{T}'$, in symbols $\mathcal{T} \overset{\mathrm{G}}{\sim} \mathcal{T}'$, if there exists a bijective function $\varphi : S \to S'$ such that:

- There is a transition $s \xrightarrow{\ell} t \in T$ with $c(\ell) = k$ iff there is a transition $\varphi(s) \xrightarrow{\ell'} \varphi(t) \in T'$ with $c'(\ell') = k$,
- $\varphi(s_0) = s'_0$, and
- $s \in S_\star$ iff $\varphi(s) \in S'_\star$.

Note: The labels of $\mathcal{T}$ and $\mathcal{T}'$ do not matter except that transitions of the same cost must be preserved.

Isomorphism vs. Graph Equivalence

- $(\sim)$ and $(\overset{\mathrm{G}}{\sim})$ are equivalence relations.
- Two isomorphic transition systems are interchangeable for all practical intents and purposes.
- Two graph-equivalent transition systems are interchangeable for most intents and purposes.
- In particular, their goal distances are identical.
- Isomorphism implies graph equivalence, but not vice versa.

D1.3 Abstraction Heuristics

Abstraction Heuristics

Definition (Abstraction Heuristic)

Let $\alpha : S \to S^\alpha$ be an abstraction of a transition system $\mathcal{T}$.

The abstraction heuristic induced by α, written h^α, is the heuristic function $h^\alpha : S \to \mathbb{R}_0^+ \cup \{\infty\}$ defined as

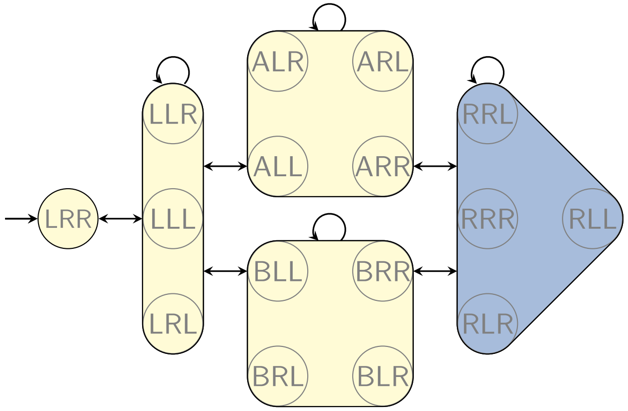
$$h^\alpha(s) = h^*_{\mathcal{T}^\alpha}(\alpha(s)) \quad \text{for all } s \in S,$$

where $h^*_{\mathcal{T}^\alpha}$ denotes the goal distance function in $\mathcal{T}^\alpha$.

Notes:

- $h^\alpha(s) = \infty$ if no goal state of $\mathcal{T}^\alpha$ is reachable from $\alpha(s)$
- We also apply abstraction terminology to planning tasks Π, which stand for their induced transition systems. For example, an abstraction of Π is an abstraction of $\mathcal{T}(\Pi)$.

Abstraction Heuristics: Example



$h^\alpha(\{p \mapsto \mathsf{L}, t_\mathsf{A} \mapsto \mathsf{R}, t_\mathsf{B} \mapsto \mathsf{R}\}) = 3$

Consistency of Abstraction Heuristics (1)

Theorem (Consistency and Admissibility of h^α)

Let α be an abstraction of a transition system $\mathcal{T}$.
Then h^α is safe, goal-aware, admissible and consistent.

Proof.

We prove goal-awareness and consistency; the other properties follow from these two.

Let $\mathcal{T} = \langle S, L, c, T, s_0, S_\star \rangle$.
Let $\mathcal{T}^\alpha = \langle S^\alpha, L, c, T^\alpha, s_0^\alpha, S_\star^\alpha \rangle$.

Goal-awareness: We need to show that $h^\alpha(s) = 0$ for all $s \in S_\star$, so let $s \in S_\star$. Then $\alpha(s) \in S_\star^\alpha$ by the definition of abstract transition systems, and hence $h^\alpha(s) = h^*_{\mathcal{T}^\alpha}(\alpha(s)) = 0$. ...

Consistency of Abstraction Heuristics (2)

Proof (continued).

Consistency: Consider any state transition $s \xrightarrow{\ell} t$ of $\mathcal{T}$.
We need to show $h^\alpha(s) \leq c(\ell) + h^\alpha(t)$.

By the definition of $\mathcal{T}^\alpha$, we get $\alpha(s) \xrightarrow{\ell} \alpha(t) \in \mathcal{T}^\alpha$.
Hence, $\alpha(t)$ is a successor of $\alpha(s)$ in $\mathcal{T}^\alpha$ via the label ℓ.

We get:

$$\begin{aligned} h^\alpha(s) &= h^*_{\mathcal{T}^\alpha}(\alpha(s)) \\ &\leq c(\ell) + h^*_{\mathcal{T}^\alpha}(\alpha(t)) \\ &= c(\ell) + h^\alpha(t), \end{aligned}$$

where the inequality holds because perfect goal distances $h^*_{\mathcal{T}^\alpha}$ are consistent in $\mathcal{T}^\alpha$.
(The shortest path from $\alpha(s)$ to the goal in $\mathcal{T}^\alpha$ cannot be longer than the shortest path from $\alpha(s)$ to the goal via $\alpha(t)$.) □

D1.4 Coarsenings and Refinements

Abstractions of Abstractions

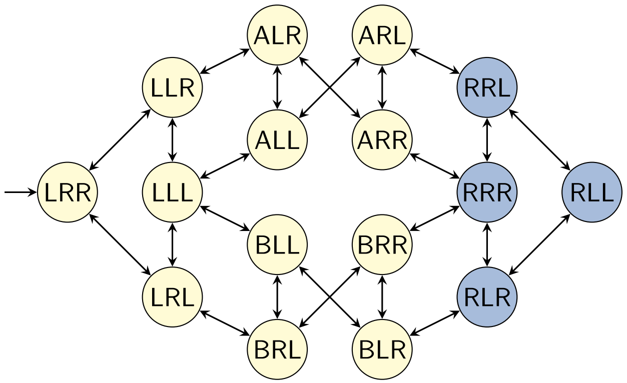
Since abstractions map transition systems to transition systems, they are composable:

- Using a first abstraction $\alpha : S \to S'$, map $\mathcal{T}$ to $\mathcal{T}^{\alpha}$.
- Using a second abstraction $\beta : S' \to S''$, map $\mathcal{T}^{\alpha}$ to $(\mathcal{T}^{\alpha})^{\beta}$.

The result is the same as directly using the abstraction $(\beta \circ \alpha)$:

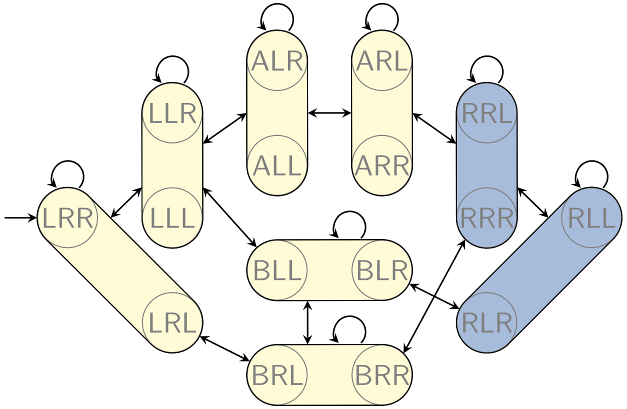
- Let $\gamma : S \to S''$ be defined as $\gamma(s) = (\beta \circ \alpha)(s) = \beta(\alpha(s))$.
- Then $\mathcal{T}^{\gamma} = (\mathcal{T}^{\alpha})^{\beta}$.

Abstractions of Abstractions: Example (1)



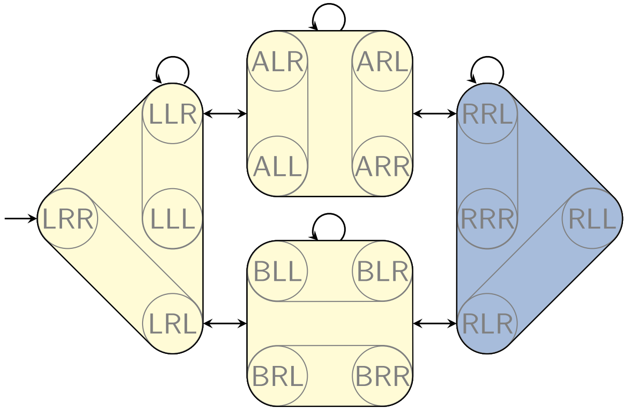
transition system $\mathcal{T}$

Abstractions of Abstractions: Example (2)



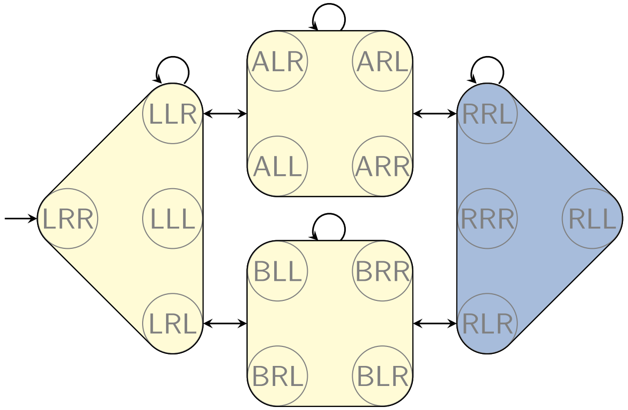
Transition system $\mathcal{T}'$ as an abstraction of $\mathcal{T}$ (ignore t_{B})

Abstractions of Abstractions: Example (3)



Transition system $\mathcal{T}''$ as an abstraction of $\mathcal{T}'$
(ignore t_A)

Abstractions of Abstractions: Example (3)



Transition system $\mathcal{T}''$ as an abstraction of $\mathcal{T}$
(ignore t_A and t_B)

Coarsenings and Refinements

Definition (Coarsening and Refinement)

Let α and γ be abstractions of the same transition system such that $\gamma = \beta \circ \alpha$ for some function β.

Then γ is called a coarsening of α and α is called a refinement of γ.

Heuristic Quality of Refinements

Theorem (Heuristic Quality of Refinements)

Let α and γ be abstractions of the same transition system such that α is a refinement of γ.

Then h^α dominates h^γ.

In other words, $h^\gamma(s) \leq h^\alpha(s) \leq h^*(s)$ for all states s.

Heuristic Quality of Refinements: Proof

Proof.

Since α is a refinement of γ,
there exists a function β with $\gamma = \beta \circ \alpha$.

For all states s of Π, we get:

$$\begin{aligned} h^{\gamma}(s) &= h^*_{\mathcal{T}^{\gamma}}(\gamma(s)) \\ &= h^*_{\mathcal{T}^{\gamma}}(\beta(\alpha(s))) \\ &= h^{\beta}_{\mathcal{T}^{\alpha}}(\alpha(s)) \\ &\leq h^*_{\mathcal{T}^{\alpha}}(\alpha(s)) \\ &= h^{\alpha}(s), \end{aligned}$$

where the inequality holds because $h^{\beta}_{\mathcal{T}^{\alpha}}$ is an admissible heuristic in the transition system $\mathcal{T}^{\alpha}$. □

D1.5 Summary

Summary

- Abstraction is one of the principled ways of deriving heuristics.
- An abstraction is a function α that maps the states S of a transition system to another (usually smaller) set S^{α}.
- This induces an abstract transition system $\mathcal{T}^{\alpha}$, which behaves like the original transition system $\mathcal{T}$ except that states mapped to the same abstract state cannot be distinguished.
- Abstractions α induce abstraction heuristics h^{α}: $h^{\alpha}(s)$ is the goal distance of $\alpha(s)$ in the abstract transition system.
- Abstraction heuristics are safe, goal-aware, admissible and consistent.
- Abstractions can be composed, leading to coarser vs. finer abstractions. Heuristics for finer abstractions dominate those for coarser ones.