

# Theory of Computer Science

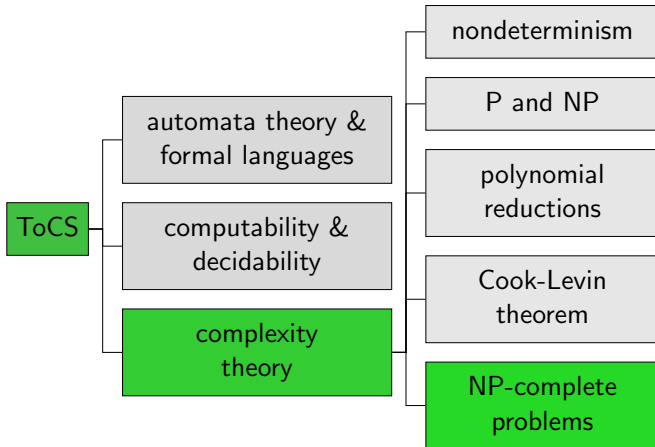
## D4. Some NP-Complete Problems, Part I

Gabriele Röger

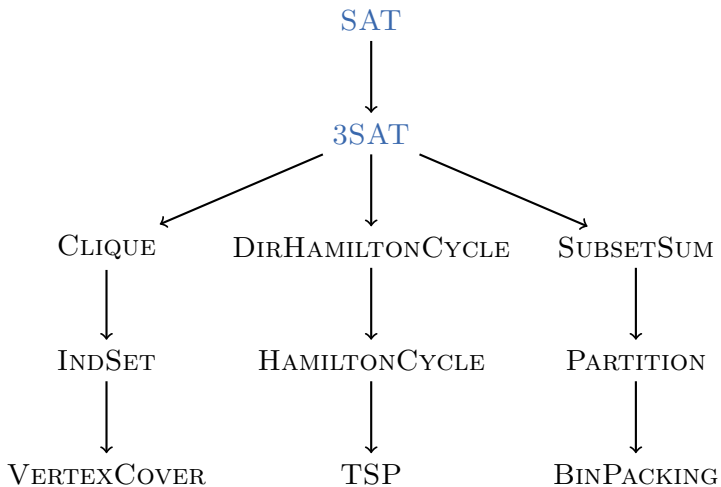
University of Basel

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# Content of the Course



# Overview of the Reductions







# CLIQUE

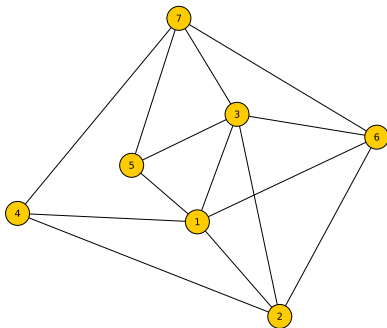
## Definition (CLIQUE)

The problem **CLIQUE** is defined as follows:

**Given:** undirected graph  $G = \langle V, E \rangle$ , number  $K \in \mathbb{N}_0$

**Question:** Does  $G$  have a clique of size at least  $K$ ,  
i. e., a set of vertices  $C \subseteq V$  with  $|C| \geq K$   
and  $\{u, v\} \in E$  for all  $u, v \in C$  with  $u \neq v$ ?

# Cliques: Exercise (slido)



How many nodes has the largest clique of this graph?



# CLIQUE is NP-Complete (1)

Theorem (CLIQUE is NP-Complete)

*CLIQUE is NP-complete.*

## CLIQUE is NP-Complete (2)

Proof.

CLIQUE  $\in$  NP: guess and check.

## CLIQUE is NP-Complete (2)

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CLIQUE is NP-hard: We show  $3\text{SAT} \leq_p \text{CLIQUE}$ .

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CLIQUE is NP-hard: We show  $3\text{SAT} \leq_p \text{CLIQUE}$ .

- We are given a 3-CNF formula  $\varphi$ , and we may assume that each clause has exactly three literals.
- In polynomial time, we must construct a graph  $G = \langle V, E \rangle$  and a number  $K$  such that:  $G$  has a clique of size at least  $K$  iff  $\varphi$  is satisfiable.

# CLIQUE is NP-Complete (2)

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↪ construction of  $V, E, K$  on the following slides.

## CLIQUE is NP-Complete (3)

Proof (continued).

Let  $m$  be the number of clauses in  $\varphi$ .

Let  $l_{ij}$  the  $j$ -th literal in clause  $i$ .

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Define  $V$ ,  $E$ ,  $K$  as follows:

# CLIQUE is NP-Complete (3)

## Proof (continued).

Let  $m$  be the number of clauses in  $\varphi$ .

Let  $\ell_{ij}$  the  $j$ -th literal in clause  $i$ .

Define  $V$ ,  $E$ ,  $K$  as follows:

- $V = \{\langle i, j \rangle \mid 1 \leq i \leq m, 1 \leq j \leq 3\}$   
     $\rightsquigarrow$  a vertex for every literal of every clause

# CLIQUE is NP-Complete (3)

## Proof (continued).

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- $V = \{\langle i, j \rangle \mid 1 \leq i \leq m, 1 \leq j \leq 3\}$   
 $\rightsquigarrow$  a vertex for every literal of every clause
- $E$  contains edge between  $\langle i, j \rangle$  and  $\langle i', j' \rangle$  if and only if
  - $i \neq i' \rightsquigarrow$  belong to **different clauses**, and
  - $l_{ij}$  and  $l_{i'j'}$  are **not complementary literals**

# CLIQUE is NP-Complete (3)

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- $K = m$

# CLIQUE is NP-Complete (3)

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$\rightsquigarrow$  obviously polynomially computable

# CLIQUE is NP-Complete (3)

## Proof (continued).

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- $K = m$

$\rightsquigarrow$  obviously polynomially computable

to show: reduction property

# CLIQUE is NP-Complete (4)

Proof (continued).

( $\Rightarrow$ ): If  $\varphi$  is satisfiable, then  $\langle V, E \rangle$  has clique of size at least  $K$ :

# CLIQUE is NP-Complete (4)

## Proof (continued).

( $\Rightarrow$ ): If  $\varphi$  is satisfiable, then  $\langle V, E \rangle$  has clique of size at least  $K$ :

- Given a satisfying variable assignment choose a vertex corresponding to a satisfied literal in each clause.
- The chosen  $K$  vertices are all connected with each other and hence form a clique of size  $K$ .

...

# CLIQUE is NP-Complete (5)

Proof (continued).

( $\Leftarrow$ ): If  $\langle V, E \rangle$  has a clique of size at least  $K$ , then  $\varphi$  is satisfiable:

# CLIQUE is NP-Complete (5)

## Proof (continued).

( $\Leftarrow$ ): If  $\langle V, E \rangle$  has a clique of size at least  $K$ , then  $\varphi$  is satisfiable:

- Consider a given clique  $C$  of size at least  $K$ .
- The vertices in  $C$  must all correspond to different clauses (vertices in the same clause are not connected by edges).

$\rightsquigarrow$  exactly one vertex per clause is included in  $C$

- Two vertices in  $C$  never correspond to complementary literals  $X$  and  $\neg X$  (due to the way we defined the edges).

## CLIQUE is NP-Complete (5)

### Proof (continued).

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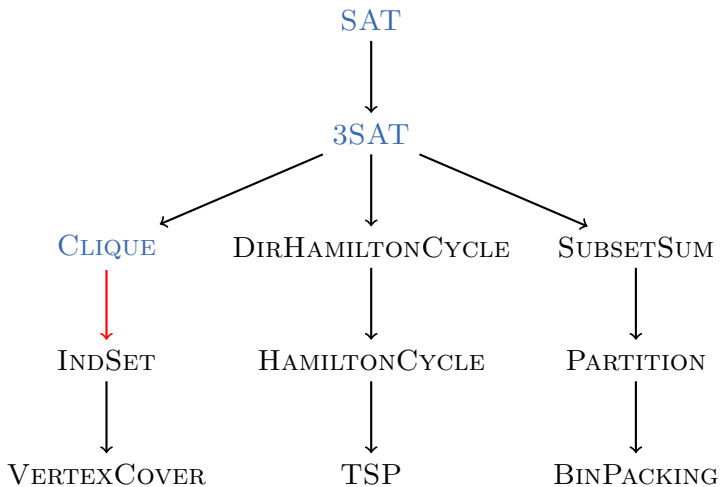
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- Two vertices in  $C$  never correspond to complementary literals  $X$  and  $\neg X$  (due to the way we defined the edges).
- If a vertex corresp. to  $X$  was chosen, map  $X$  to T (true).
- If a vertex corresp. to  $\neg X$  was chosen, map  $X$  to F (false).
- If neither was chosen, arbitrarily map  $X$  to T or F.

$\rightsquigarrow$  satisfying assignment



# CLIQUE $\leq_p$ INDSET



# INDSET

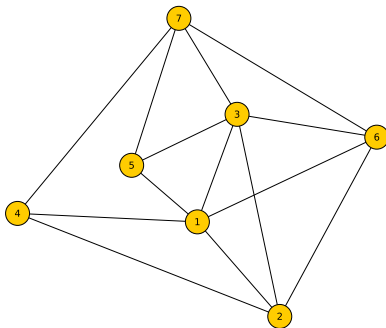
## Definition (INDSET)

The problem **INDSET** is defined as follows:

**Given:** undirected graph  $G = \langle V, E \rangle$ , number  $K \in \mathbb{N}_0$

**Question:** Does  $G$  have an independent set of size at least  $K$ ,  
i. e., a set of vertices  $I \subseteq V$  with  $|I| \geq K$   
and  $\{u, v\} \notin E$  for all  $u, v \in I$  with  $u \neq v$ ?

# Independent Set: Exercise (slido)



Does this graph have an independent set of size 3?



# INDSET is NP-Complete (1)

Theorem (INDSET is NP-Complete)

INDSET *is NP-complete.*

# INDSET is NP-Complete (1)

Theorem (INDSET is NP-Complete)

INDSET is NP-complete.

Proof.

INDSET  $\in$  NP: guess and check. ...

## INDSET is NP-Complete (2)

Proof (continued).

INDSET is NP-hard: We show  $\text{CLIQUE} \leq_p \text{INDSET}$ .

## INDSET is NP-Complete (2)

Proof (continued).

**INDSET is NP-hard:** We show  $\text{CLIQUE} \leq_p \text{INDSET}$ .

We describe a polynomial reduction  $f$ .

Let  $\langle G, K \rangle$  with  $G = \langle V, E \rangle$  be the given input for **CLIQUE**.

## INDSET is NP-Complete (2)

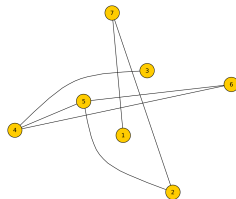
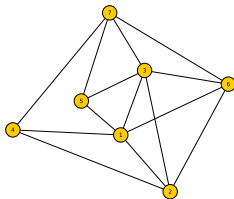
Proof (continued).

**INDSET is NP-hard:** We show  $\text{CLIQUE} \leq_p \text{INDSET}$ .

We describe a polynomial reduction  $f$ .

Let  $\langle G, K \rangle$  with  $G = \langle V, E \rangle$  be the given input for **CLIQUE**.

Then  $f(\langle G, K \rangle)$  is the **INDSET** instance  $\langle \bar{G}, K \rangle$ , where  $\bar{G} := \langle V, \bar{E} \rangle$  and  $\bar{E} := \{\{u, v\} \subseteq V \mid u \neq v, \{u, v\} \notin E\}$ .  
(This graph  $\bar{G}$  is called the **complement graph** of  $G$ .)



## INDSET is NP-Complete (2)

Proof (continued).

**INDSET is NP-hard:** We show  $\text{CLIQUE} \leq_p \text{INDSET}$ .

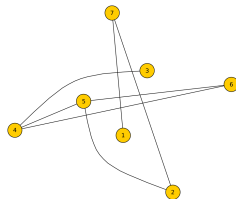
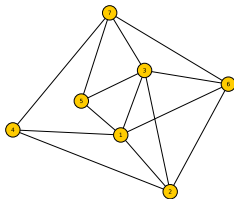
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(This graph  $\bar{G}$  is called the **complement graph** of  $G$ .)

Clearly  $f$  can be computed in polynomial time.

...



# INDSET is NP-Complete (3)

## Proof (continued).

We have:

$\langle \langle V, E \rangle, K \rangle \in \text{CLIQUE}$

iff there exists a set  $V' \subseteq V$  with  $|V'| \geq K$   
and  $\{u, v\} \in E$  for all  $u, v \in V'$  with  $u \neq v$

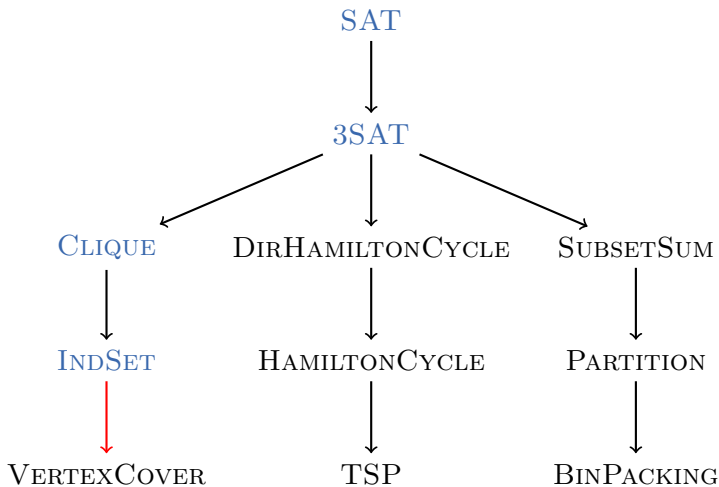
iff there exists a set  $V' \subseteq V$  with  $|V'| \geq K$   
and  $\{u, v\} \notin \bar{E}$  for all  $u, v \in V'$  with  $u \neq v$

iff  $\langle \langle V, \bar{E} \rangle, K \rangle \in \text{INDSET}$

iff  $f(\langle \langle V, E \rangle, K \rangle) \in \text{INDSET}$

and hence  $f$  is a reduction. □

# INDSET $\leq_p$ VERTEXCOVER



# VERTEXCOVER

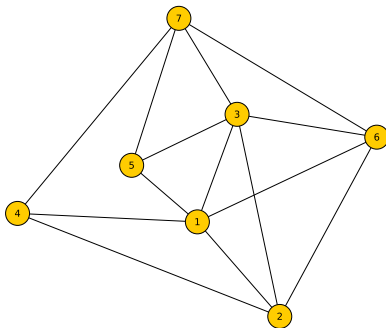
## Definition (VERTEXCOVER)

The problem **VERTEXCOVER** is defined as follows:

**Given:** undirected graph  $G = \langle V, E \rangle$ , number  $K \in \mathbb{N}_0$

**Question:** Does  $G$  have a vertex cover of size at most  $K$ , i. e., a set of vertices  $C \subseteq V$  with  $|C| \leq K$  and  $\{u, v\} \cap C \neq \emptyset$  for all  $\{u, v\} \in E$ ?

# Vertex Cover: Exercise (slido)



Does this graph have a vertex cover of size 4?



# VERTEXCOVER is NP-Complete (1)

Theorem (VERTEXCOVER is NP-Complete)

VERTEXCOVER is *NP-complete*.

# VERTEXCOVER is NP-Complete (2)

Proof.

VERTEXCOVER  $\in$  NP: guess and check.

## VERTEXCOVER is NP-Complete (2)

### Proof.

VERTEXCOVER  $\in$  NP: guess and check.

VERTEXCOVER is NP-hard:

We show  $\text{INDSET} \leq_p \text{VERTEXCOVER}$ .

## VERTEXCOVER is NP-Complete (2)

### Proof.

VERTEXCOVER  $\in$  NP: guess and check.

VERTEXCOVER is NP-hard:

We show  $\text{INDSET} \leq_p \text{VERTEXCOVER}$ .

We describe a polynomial reduction  $f$ .

Let  $\langle G, K \rangle$  with  $G = \langle V, E \rangle$  be the given input for INDSET.

## VERTEXCOVER is NP-Complete (2)

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VERTEXCOVER  $\in$  NP: guess and check.

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We describe a polynomial reduction  $f$ .

Let  $\langle G, K \rangle$  with  $G = \langle V, E \rangle$  be the given input for INDSET.

Then  $f(\langle G, K \rangle) := \langle G, |V| - K \rangle$ .

This can clearly be computed in polynomial time.

...

## VERTEXCOVER is NP-Complete (3)

Proof (continued).

For vertex set  $V' \subseteq V$ , we write  $\overline{V'}$  for its **complement**  $V \setminus V'$ .

## VERTEXCOVER is NP-Complete (3)

Proof (continued).

For vertex set  $V' \subseteq V$ , we write  $\overline{V'}$  for its **complement**  $V \setminus V'$ .

**Observation:** a set of vertices is a vertex cover  
iff its complement is an independent set.

## VERTEXCOVER is NP-Complete (3)

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For vertex set  $V' \subseteq V$ , we write  $\overline{V'}$  for its **complement**  $V \setminus V'$ .

**Observation:** a set of vertices is a vertex cover  
iff its complement is an independent set.

We thus have:

$$\langle \langle V, E \rangle, K \rangle \in \text{INDSET}$$

iff  $\langle V, E \rangle$  has an independent set  $I$  with  $|I| \geq K$

iff  $\langle V, E \rangle$  has a vertex cover  $C$  with  $|\overline{C}| \geq K$

iff  $\langle V, E \rangle$  has a vertex cover  $C$  with  $|C| \leq |V| - K$

iff  $\langle \langle V, E \rangle, |V| - K \rangle \in \text{VERTEXCOVER}$

iff  $f(\langle \langle V, E \rangle, K \rangle) \in \text{VERTEXCOVER}$

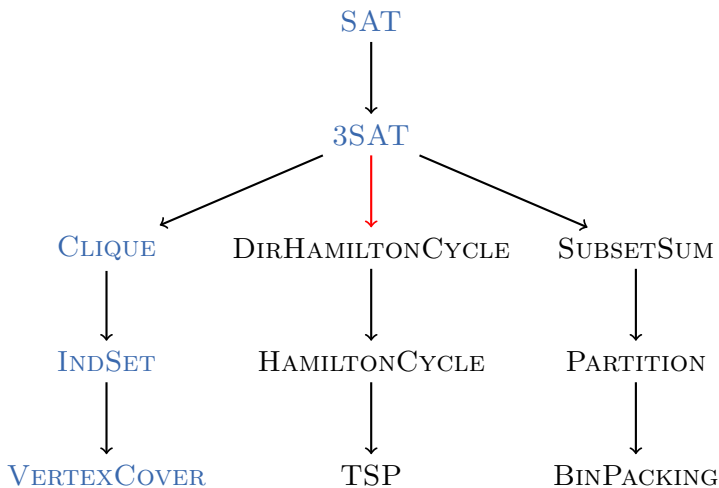
and hence  $f$  is a reduction. □

# Questions



Questions?

# Routing Problems

$3\text{SAT} \leq_p \text{DIRHAMILTONCYCLE}$ 

# DIRHAMILTONCYCLE is NP-Complete (1)

## Definition (Reminder: DIRHAMILTONCYCLE)

The problem **DIRHAMILTONCYCLE** is defined as follows:

**Given:** directed graph  $G = \langle V, E \rangle$

**Question:** Does  $G$  contain a Hamilton cycle?

# DIRHAMILTONCYCLE is NP-Complete (1)

## Definition (Reminder: DIRHAMILTONCYCLE)

The problem **DIRHAMILTONCYCLE** is defined as follows:

**Given:** directed graph  $G = \langle V, E \rangle$

**Question:** Does  $G$  contain a Hamilton cycle?

## Theorem

**DIRHAMILTONCYCLE** is NP-complete.

# DIRHAMILTONCYCLE is NP-Complete (2)

Proof.

DIRHAMILTONCYCLE  $\in$  NP: guess and check.

# DIRHAMILTONCYCLE is NP-Complete (2)

## Proof.

DIRHAMILTONCYCLE  $\in$  NP: guess and check.

DIRHAMILTONCYCLE is NP-hard:

We show  $3\text{SAT} \leq_p \text{DIRHAMILTONCYCLE}$ .

# DIRHAMILTONCYCLE is NP-Complete (2)

## Proof.

DIRHAMILTONCYCLE  $\in$  NP: guess and check.

DIRHAMILTONCYCLE is NP-hard:

We show  $3SAT \leq_p$  DIRHAMILTONCYCLE.

- We are given a 3-CNF formula  $\varphi$  where each clause contains exactly three literals and no clause contains duplicated literals.
- We must, in polynomial time, construct a directed graph  $G = \langle V, E \rangle$  such that:  
 $G$  contains a Hamilton cycle iff  $\varphi$  is satisfiable.
- construction of  $\langle V, E \rangle$  on the following slides

# DIRHAMILTONCYCLE is NP-Complete (3)

## Proof (continued).

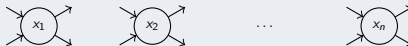
- Let  $X_1, \dots, X_n$  be the atomic propositions in  $\varphi$ .
- Let  $c_1, \dots, c_m$  be the clauses of  $\varphi$  with  $c_i = (l_{i1} \vee l_{i2} \vee l_{i3})$ .
- Construct a graph with  $6m + n$  vertices (described on the following slides).

...

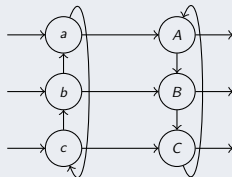
# DIRHAMILTONCYCLE is NP-Complete (4)

## Proof (continued).

- For every variable  $X_i$ , add vertex  $x_i$  with 2 incoming and 2 outgoing edges:



- For every clause  $c_j$ , add the subgraph  $C_j$  with 6 vertices:



- We describe later how to connect these parts.

# DIRHAMILTONCYCLE is NP-Complete (5)

## Proof (continued).

Let  $\pi$  be a Hamilton cycle of the total graph.

- Whenever  $\pi$  enters subgraph  $C_j$  from one of its “entrances”, it must leave via the corresponding “exit”:  
( $a \rightarrow A, b \rightarrow B, c \rightarrow C$ ).  
Otherwise,  $\pi$  cannot be a Hamilton cycle.
- Hamilton cycles can behave in the following ways with regard to  $C_j$ :
  - $\pi$  passes through  $C_j$  once (from any entrance)
  - $\pi$  passes through  $C_j$  twice (from any two entrances)
  - $\pi$  passes through  $C_j$  three times (once from every entrance)

# DIRHAMILTONCYCLE is NP-Complete (6)

## Proof (continued).

Connect the “open ends” in the graph as follows:

- Identify entrances/exits of the clause subgraph  $C_j$  with the three literals in clause  $c_j$ .
- One exit of  $x_i$  is **positive**, the other one is **negative**.
- For the **positive** exit, determine the clauses in which the positive literal  $X_i$  occurs:
  - Connect the positive exit of  $x_i$  with the  $X_i$ -entrance of the first such clause graph.
  - Connect the  $X_i$ -exit of this clause graph with the  $X_i$ -entrance of the second such clause graph, and so on.
  - Connect the  $X_i$ -exit of the last such clause graph with the positive entrance of  $x_{i+1}$  (or  $x_1$  if  $i = n$ ).
- analogously for the **negative** exit of  $x_i$  and the literal  $\neg X_i$

# DIRHAMILTONCYCLE is NP-Complete (7)

## Proof (continued).

The construction is polynomial and is a reduction:

( $\Rightarrow$ ): **construct a Hamilton cycle from a satisfying assignment**

- Given a satisfying assignment  $\mathcal{I}$ , construct a Hamilton cycle that leaves  $x_i$  through the positive exit if  $\mathcal{I}(X_i)$  is true and by the negative exit if  $\mathcal{I}(X_i)$  is false.
- Afterwards, we visit all  $C_j$ -subgraphs for clauses that are satisfied by this literal.
- In total, we visit each  $C_j$ -subgraph 1–3 times.

# DIRHAMILTONCYCLE is NP-Complete (8)

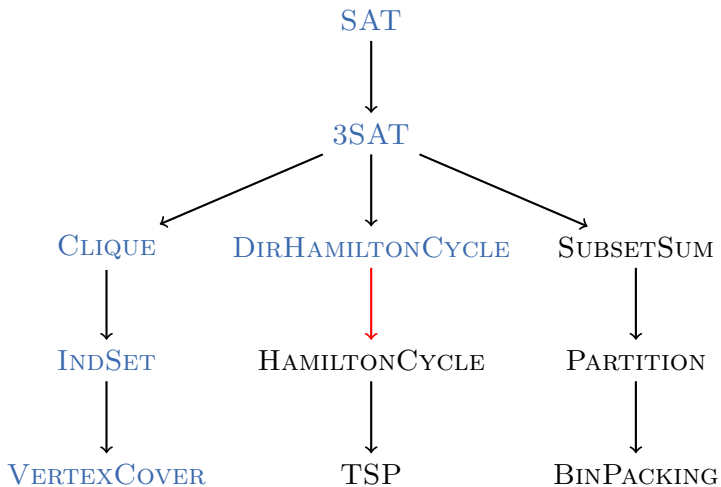
Proof (continued).

( $\Leftarrow$ ): **construct a satisfying assignment from a Hamilton cycle**

- A Hamilton cycle visits every vertex  $x_i$  and leaves it by the positive or negative exit.
- Map  $X_i$  to true or false depending on which exit is used to leave  $x_i$ .
- Because the cycle must traverse each  $C_j$ -subgraph at least once (otherwise it is not a Hamilton cycle), this results in a satisfying assignment. (Details omitted.)



# DIRHAMILTONCYCLE $\leq_p$ HAMILTONCYCLE



# HAMILTONCYCLE is NP-Complete (1)

## Definition (Reminder: HAMILTONCYCLE)

The problem **HAMILTONCYCLE** is defined as follows:

**Given:** undirected graph  $G = \langle V, E \rangle$

**Question:** Does  $G$  contain a Hamilton cycle?

# HAMILTONCYCLE is NP-Complete (1)

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## Theorem

**HAMILTONCYCLE** is NP-complete.

# HAMILTONCYCLE is NP-Complete (2)

Proof sketch.

HAMILTONCYCLE  $\in$  NP: guess and check.

# HAMILTONCYCLE is NP-Complete (2)

Proof sketch.

HAMILTONCYCLE  $\in$  NP: guess and check.

HAMILTONCYCLE is NP-hard: We show  
DIRHAMILTONCYCLE  $\leq_p$  HAMILTONCYCLE.

# HAMILTONCYCLE is NP-Complete (2)

## Proof sketch.

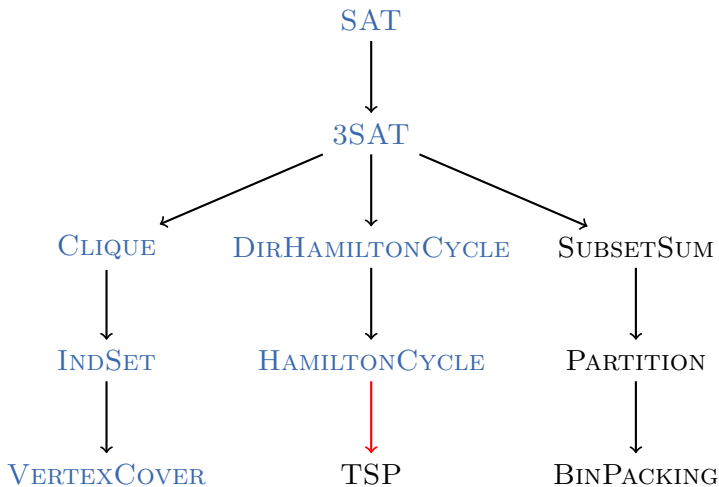
HAMILTONCYCLE  $\in$  NP: guess and check.

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DIRHAMILTONCYCLE  $\leq_p$  HAMILTONCYCLE.

Basic building block of the reduction:



# HAMILTONCYCLE $\leq_p$ TSP



# TSP is NP-Complete (1)

## Definition (Reminder: TSP)

**TSP** (traveling salesperson problem) is the following decision problem:

- **Given:** finite set  $S \neq \emptyset$  of cities, symmetric cost function  $cost : S \times S \rightarrow \mathbb{N}_0$ , cost bound  $K \in \mathbb{N}_0$
- **Question:** Is there a tour with total cost at most  $K$ , i. e., a permutation  $\langle s_1, \dots, s_n \rangle$  of the cities with 
$$\sum_{i=1}^{n-1} cost(s_i, s_{i+1}) + cost(s_n, s_1) \leq K?$$

## Theorem

TSP is NP-complete.

## TSP is NP-Complete (2)

### Proof.

TSP  $\in$  NP: guess and check.

TSP is NP-hard: We showed  $\text{HAMILTONCYCLE} \leq_p \text{TSP}$   
in Chapter D2. □

# Questions



Questions?

# Summary

# Summary

- In this chapter we showed NP-completeness of
  - three classical graph problems:  
CLIQUE, INDSET, VERTEXCOVER
  - three classical routing problems:  
DIRHAMILTONCYCLE, HAMILTONCYCLE, TSP