

Theory of Computer Science

B13. Type-1 and Type-0 Languages: Closure & Decidability

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April 8, 2026

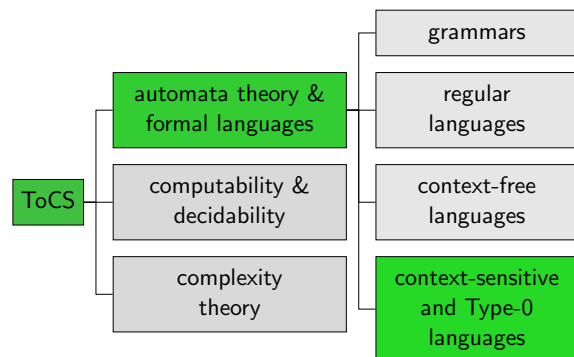
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B13.1 Turing Machines vs. Grammars

B13.2 Closure Properties and Decidability

Content of the Course



B13.1 Turing Machines vs. Grammars

Turing Machines

We have seen several variants of Turing machines:

- ▶ Deterministic TM with head movements left or right
- ▶ Deterministic TM with head movements left, right or **neutral**
- ▶ **Multitape** Turing machines
- ▶ **Nondeterministic** Turing machines

All variants recognize the same languages.

We mentioned earlier that we can relate Turing machines to the **Type-1 and Type-0 languages**.

Reminder: Context-sensitive Grammar

Type-1 languages are also called **context-sensitive** languages.

Definition (Context-sensitive Grammar)

A **context-sensitive grammar** is a 4-tuple $\langle V, \Sigma, R, S \rangle$ with

- ▶ V finite set of variables (nonterminal symbols)
- ▶ Σ finite alphabet of terminal symbols with $V \cap \Sigma = \emptyset$
- ▶ $R \subseteq (V \cup \Sigma)^* V (V \cup \Sigma)^* \times (V \cup \Sigma)^*$ finite set of rules, where all rules are of the form $\alpha B \gamma \rightarrow \alpha \beta \gamma$ with $B \in V$ and $\alpha, \gamma \in (V \cup \Sigma)^*$ and $\beta \in (V \cup \Sigma)^+$.
Exception: $S \rightarrow \varepsilon$ is allowed if S never occurs on the right-hand side of a rule.
- ▶ $S \in V$ start variable.

One Automata Model for Two Grammar Types?

Don't we need different automata models for context-sensitive and Type-0 languages?



Picture courtesy of stockimages / FreeDigitalPhotos.net

Linear Bounded Automata: Idea

- ▶ **Linear bounded automata** are NTMs that may only use the **part of the tape occupied by the input word**.
- ▶ one way of formalizing this: NTMs where blank symbol may never be replaced by a different symbol

Linear Bounded Turing Machines: Definition

Definition (Linear Bounded Automata)

An NTM $M = \langle Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}} \rangle$ is called a **linear bounded automaton (LBA)** if for all $q \in Q \setminus \{q_{\text{accept}}, q_{\text{reject}}\}$ and all transition rules $\langle q', c, y \rangle \in \delta(q, \square)$ we have $c = \square$.

German: linear beschränkte Turingmaschine

LBAs Recognize Type-1 Languages

Theorem

The languages that can be recognized by linear bounded automata are exactly the context-sensitive (type-1) languages.

Without proof.

proof sketch for grammar \Rightarrow NTM direction:

- ▶ computation of the NTM follows the production of the word in the grammar **in opposite order**
- ▶ accept when only the start symbol (and blanks) are left on the tape
- ▶ because the language is context-sensitive, we never need additional space on the tape (empty word needs special treatment)

NTMs Recognize Type-0 Languages

Theorem

The languages that can be recognized by nondeterministic Turing machines are exactly the type-0 languages.

Without proof.

proof sketch for grammar \Rightarrow NTM direction:

- ▶ analogous to previous proof
- ▶ for grammar rules $w_1 \rightarrow w_2$ with $|w_1| > |w_2|$, we must “insert” symbols into the existing tape content; this is a bit tedious, but not very difficult

What about the Deterministic Variants?

We know that DTMs and NTMs recognize the same languages.
Hence:

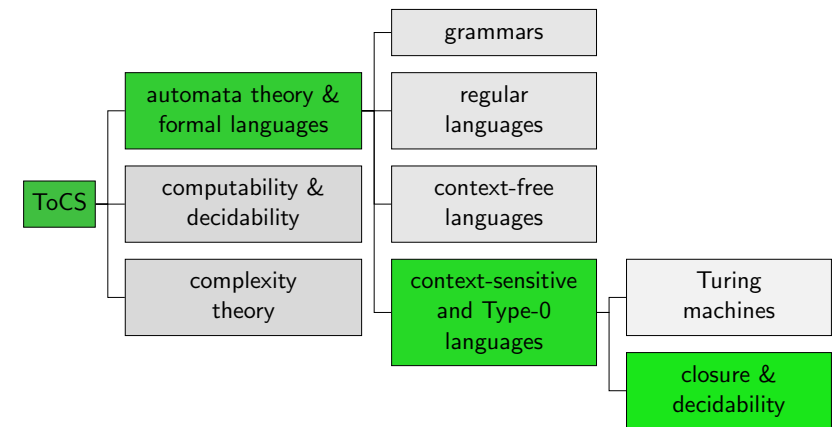
Corollary

*The **Turing-recognizable** languages are exactly the **Type-0** languages.*

Note: It is an open problem whether **deterministic** LBAs can recognize exactly the type-1 languages.

B13.2 Closure Properties and Decidability

Content of the Course



Closure Properties

	Intersection	Union	Complement	Concatenation	Star
Type 3	Yes	Yes	Yes	Yes	Yes
Type 2	No	Yes	No	Yes	Yes
Type 1	Yes ⁽²⁾	Yes ⁽¹⁾	Yes ⁽²⁾	Yes ⁽¹⁾	Yes ⁽¹⁾
Type 0	Yes ⁽²⁾	Yes ⁽¹⁾	No ⁽³⁾	Yes ⁽¹⁾	Yes ⁽¹⁾

Proofs?

- (1) proof via grammars, similar to context-free cases
- (2) without proof
- (3) proof in later chapters (part C)

Decidability

	Word problem	Emptiness problem	Equivalence problem	Intersection problem
Type 3	Yes	Yes	Yes	Yes
Type 2	Yes	Yes	No	No
Type 1	Yes ⁽¹⁾	No ⁽³⁾	No ⁽²⁾	No ⁽²⁾
Type 0	No ⁽⁴⁾	No ⁽⁴⁾	No ⁽²⁾	No ⁽²⁾

Proofs?

- (1) same argument we used for context-free languages
- (2) because already undecidable for context-free languages
- (3) without proof
- (4) proofs in later chapters (part C)

Summary

- ▶ **Turing machines** recognize exactly the **type-0** languages.
- ▶ **Linear bounded automata** recognize exactly the **context-sensitive** languages.
- ▶ The context-sensitive and type-0 languages are **closed** under **almost all** usual operations.
 - ▶ exception: **type-0 not closed** under **complement**
- ▶ For context-sensitive and type-0 languages **almost no problem is decidable**.
 - ▶ exception: **word problem** for **context-sensitive** lang. decidable

What's Next?

contents of this course:

- background** ✓
 - ▷ mathematical foundations and proof techniques
- automata theory and formal languages** ✓
 - ▷ What is a computation?
- Turing computability**
 - ▷ What can be computed at all?
- complexity theory**
 - ▷ What can be computed efficiently?
- more computability theory**
 - ▷ Other models of computability