Theory of Computer Science

B8. Context-free Languages: ε-Rules & Chomsky Normal Form

Gabriele Röger

University of Basel

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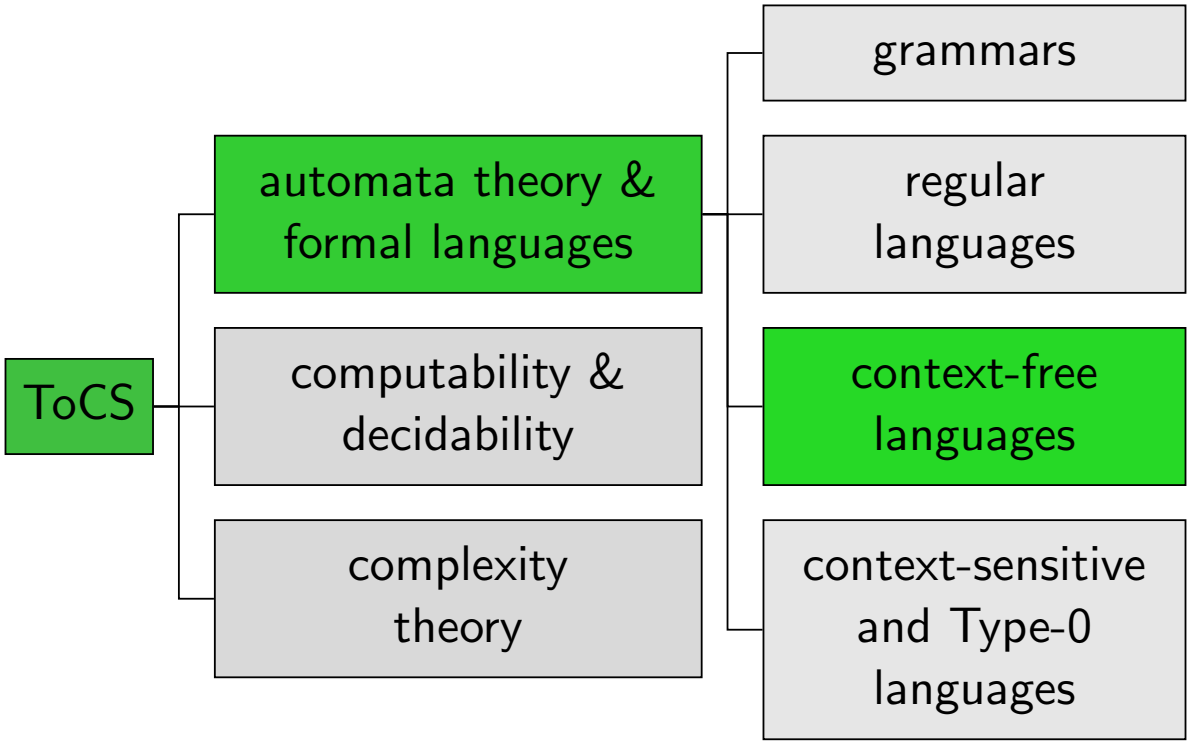
Content of the Course



Context-free Grammars and ε-Rules

Repetition: Context-free Grammars

Definition (Context-free Grammar)

A context-free grammar is a 4-tuple $\langle V, \Sigma, P, S \rangle$ with

1. V finite set of variables,
2. Σ finite alphabet of terminal symbols (with $V \cap \Sigma = \emptyset$),
3. $P \subseteq (V \times (V \cup \Sigma)^+) \cup \{\langle S, \varepsilon \rangle\}$ finite set of rules,
4. If $S \to \varepsilon \in P$, then all other rules in $V \times ((V \setminus \{S\}) \cup \Sigma)^+$.
5. $S \in V$ start variable.

- All rules have a single variable on the left hand side.
- If the right hand side of a rule is empty (ε), then the left hand side is the start variable.
- If there is a rule with an empty right hand side, then the start variable does not occur on the right hand side of any rule.

Context-free Grammars: Exercise

We have used the pumping lemma for regular languages to show that $L = \{a^n b^n \mid n \in \mathbb{N}_0\}$ is not regular.

Show that it is context-free by specifying a suitable grammar G with $\mathcal{L}(G) = L$.

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- If the right hand side of a rule is empty (ε), then the left hand side is the start variable.
- If there is a rule with an empty right hand side, then the start variable does not occur on the right hand side of any rule.

With regular grammars, these restrictions could be lifted.
How about context-free grammars?

Reminder: Start Variable in Right-Hand Side of Rules

For every type-0 language L there is a grammar where the start variable does not occur on the right-hand side of any rule.

Theorem

For every grammar $G = \langle V, \Sigma, P, S \rangle$ there is a grammar $G' = \langle V', \Sigma, P', S \rangle$ with rules $P' \subseteq (V' \cup \Sigma)^+ \times (V' \setminus \{S\} \cup \Sigma)^*$ such that $\mathcal{L}(G) = \mathcal{L}(G')$.

In the proof we constructed a suitable grammar, where the rules in P' were not fundamentally different from the rules in P:

- for rules from $V \times (V \cup \Sigma)^+$, we only introduced additional rules from $V' \times (V' \cup \Sigma)^+$, and
- for rules from $V \times \varepsilon$, we only introduced rules from $V' \times \varepsilon$,

where $V' = V \cup \{S'\}$ for some new variable $S' \notin V$.

ε-Rules

Theorem

For every grammar G with rules $P \subseteq V \times (V \cup \Sigma)^*$ there is a context-free grammar G' with $\mathcal{L}(G) = \mathcal{L}(G')$.

ε-Rules

Theorem

For every grammar G with rules $P \subseteq V \times (V \cup \Sigma)^*$ there is a context-free grammar G' with $\mathcal{L}(G) = \mathcal{L}(G')$.

Proof.

Let $G = \langle V, \Sigma, P, S \rangle$ be a grammar with $P \subseteq V \times (V \cup \Sigma)^*$.

Let $G' = \langle V', \Sigma, P', S \rangle$ be a grammar with $\mathcal{L}(G) = \mathcal{L}(G')$ with $P' \subseteq V' \times ((V' \setminus S) \cup \Sigma)^*$.

Let $V_\varepsilon = \{A \in V' \mid A \Rightarrow^*_{G'} \varepsilon\}$. We can find this set V_ε by first collecting all variables A with rule $A \to \varepsilon \in P'$ and then successively adding additional variables B if there is a rule $B \to A_1 A_2 \dots A_k \in P'$ and the variables A_i are already in the set for all $1 \leq i \leq k$. ...

ε-Rules

Theorem

For every grammar G with rules $P \subseteq V \times (V \cup \Sigma)^*$ there is a context-free grammar G' with $\mathcal{L}(G) = \mathcal{L}(G')$.

Proof (continued).

Let P'' be the rule set that is constructed from P' by

- adding rules that obviate the need for $A \to \varepsilon$ rules: for every existing rule $B \to w$ with $B \in V', w \in (V' \cup \Sigma)^+$, let I_ε be the set of positions where w contains a variable $A \in V_\varepsilon$. For every non-empty set $I' \subseteq I_\varepsilon$, add a new rule $B \to w'$, where w' is constructed from w by removing the variables at all positions in I'.
- removing all rules of the form $A \to \varepsilon$ ($A \neq S$).

Then $G'' = \langle V', \Sigma, P'', S \rangle$ is context-free and $\mathcal{L}(G) = \mathcal{L}(G'')$. □

Example

Consider $G = \langle \{\mathsf{X}, \mathsf{Y}, \mathsf{Z}, \mathsf{S}\}, \{\mathtt{a}, \mathtt{b}\}, R, S \rangle$ with rules:

$$\mathsf{S} \to \varepsilon \mid \mathsf{XY}$$
$$\mathsf{X} \to \mathtt{a}\mathsf{XY}\mathtt{b}\mathsf{X} \mid \mathsf{YZ}$$
$$\mathsf{Y} \to \varepsilon \mid \mathtt{b}$$
$$\mathsf{Z} \to \varepsilon \mid \mathtt{a}$$

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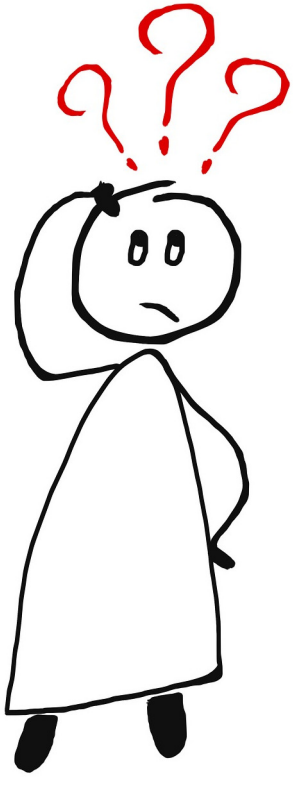
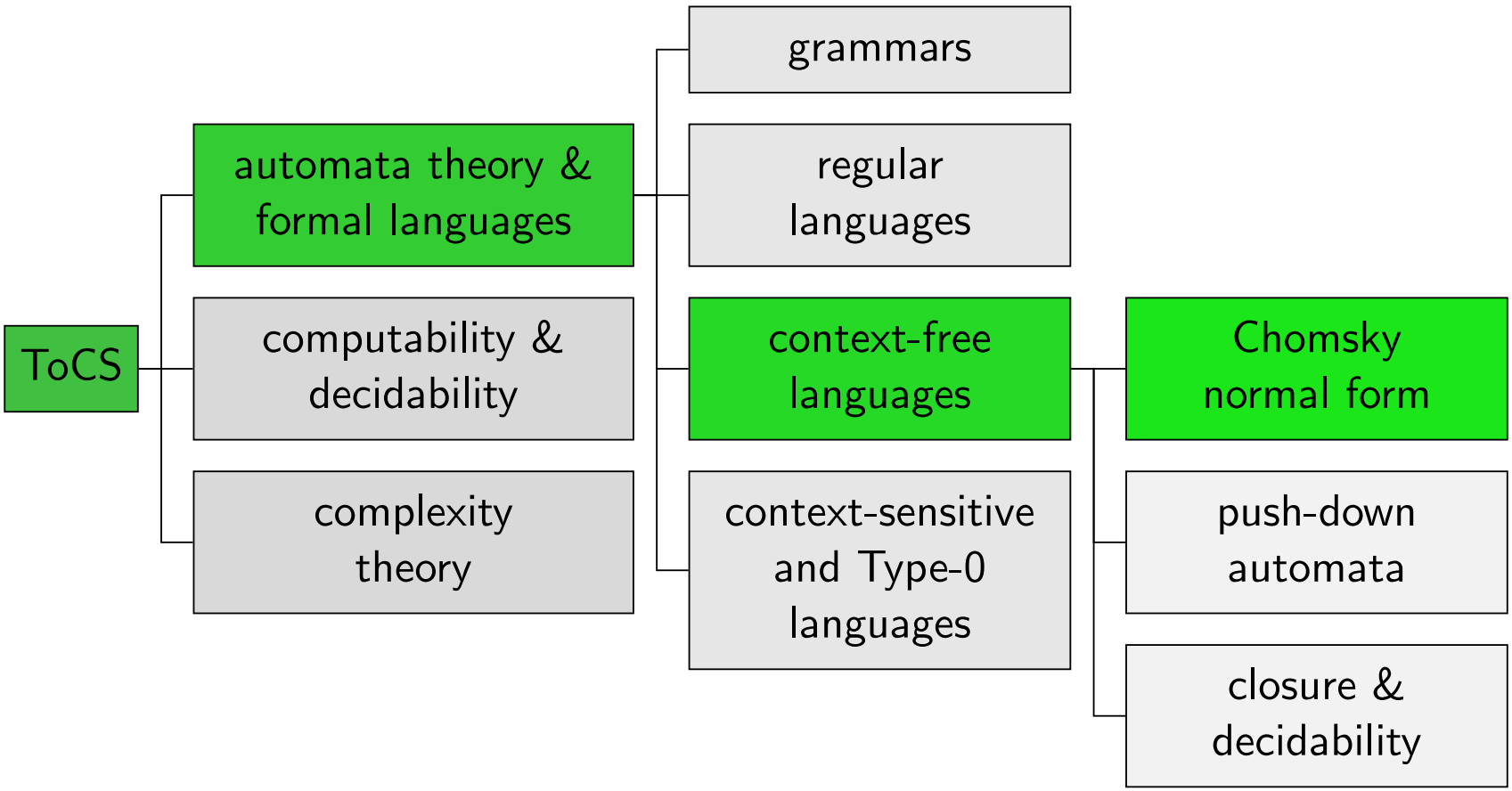
Questions



Questions?

Chomsky Normal Form

Content of the Course



Chomsky Normal Form: Motivation

As in logical formulas (and other kinds of structured objects), normal forms for grammars are useful:

- they show which aspects are critical for defining grammars and which ones are just syntactic sugar
- they allow proofs and algorithms to be restricted to a limited set of grammars (inputs): those in normal form

Hence we now consider a normal form for context-free grammars.

Chomsky Normal Form: Definition

Definition (Chomsky Normal Form)

A context-free grammar G is in Chomsky normal form (CNF) if all rules have one of the following three forms:

- $A \to BC$ with variables A, B, C, or
- $A \to a$ with variable A, terminal symbol a, or
- $S \to \varepsilon$ with start variable S.

in short:

rule set $P \subseteq (V \times (V'V' \cup \Sigma)) \cup \{\langle S, \varepsilon \rangle\}$ with $V' = V \setminus \{S\}$

German: Chomsky-Normalform

Chomsky Normal Form: Theorem

Theorem

For every context-free grammar G there is a context-free grammar G' in Chomsky normal form with $\mathcal{L}(G) = \mathcal{L}(G')$.

Chomsky Normal Form: Theorem

Theorem

For every context-free grammar G there is a context-free grammar G' in Chomsky normal form with $\mathcal{L}(G) = \mathcal{L}(G')$.

Proof.

The following algorithm converts the rule set of G into CNF:

Step 1: Eliminate rules of the form $A \to B$ with variables A, B.

If there are sets of variables $\{B_1, \ldots, B_k\}$ with rules $B_1 \to B_2, B_2 \to B_3, \ldots, B_{k-1} \to B_k, B_k \to B_1$, then replace these variables by a new variable B.

Define a strict total order $<$ on the variables such that $A \to B \in P$ implies that $A < B$. Iterate from the largest to the smallest variable A and eliminate all rules of the form $A \to B$ while adding rules $A \to w$ for every rule $B \to w$ with $w \in (V \cup \Sigma)^+$. ...

Chomsky Normal Form: Theorem

Theorem

For every context-free grammar G there is a context-free grammar G' in Chomsky normal form with $\mathcal{L}(G) = \mathcal{L}(G')$.

Proof (continued).

Step 2: Eliminate rules with terminal symbols on the right-hand side that do not have the form $A \to a$.

For every terminal symbol $a \in \Sigma$ add a new variable A_a and the rule $A_a \to a$.

Replace all terminal symbols in all rules that do not have the form $A \to a$ with the corresponding newly added variables. ...

Chomsky Normal Form: Theorem

Theorem

For every context-free grammar G there is a context-free grammar G' in Chomsky normal form with $\mathcal{L}(G) = \mathcal{L}(G')$.

Proof (continued).

Step 3: Eliminate rules of the form $A \to B_1 B_2 \dots B_k$ with $k > 2$

For every rule of the form $A \to B_1 B_2 \dots B_k$ with $k > 2$, add new variables $C_2, \dots, C_{k-1}$ and replace the rule with

$$\begin{aligned} A &\to B_1 C_2 \\ C_2 &\to B_2 C_3 \\ &\vdots \\ C_{k-1} &\to B_{k-1} B_k \end{aligned}$$

Example

Consider $G = \langle \{\mathsf{Y}, \mathsf{Z}, \mathsf{S}\}, \{\mathtt{a}, \mathtt{b}\}, R, S \rangle$ with rules:

$$\mathsf{S} \to \mathtt{a}\mathsf{Z}\mathtt{b}\mathsf{Y} \mid \mathsf{Y} \mid \mathtt{ab}$$
$$\mathsf{Y} \to \mathsf{Z} \mid \mathtt{b}$$
$$\mathsf{Z} \to \mathsf{Y} \mid \mathtt{b}\mathsf{S}\mathtt{a}$$

$\to$ blackboard

Chomsky Normal Form: Length of Derivations

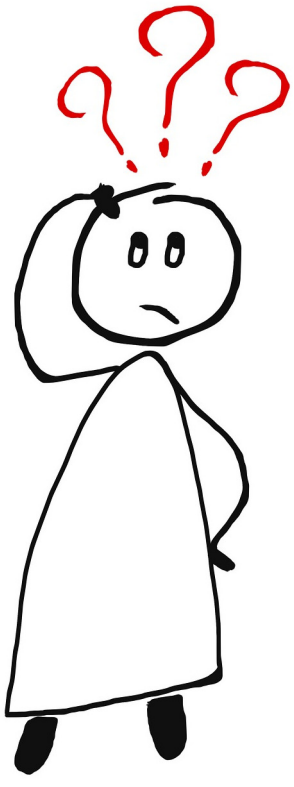
Observation

Let G be a grammar in Chomsky normal form,
and let $w \in \mathcal{L}(G)$ be a non-empty word generated by G.

Then all derivations of w have exactly $2|w| - 1$ derivation steps.

Why?

Questions



Questions?

Summary

Summary

- The restriction of ε-occurrences in rules is not necessary to characterize the set of context-free languages.
- Every context-free language has a grammar in Chomsky normal form. All rules have form
  - $A \to BC$ with variables A, B, C, or
  - $A \to a$ with variable A, terminal symbol a, or
  - $S \to \varepsilon$ with start variable S.