

Theory of Computer Science

B8. Context-free Languages: ε -Rules & Chomsky Normal Form

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Theory of Computer Science

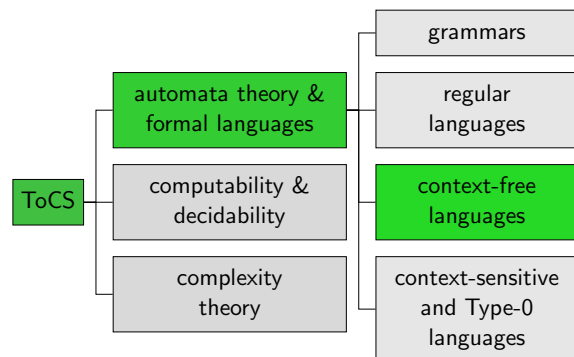
March 23, 2026 — B8. Context-free Languages: ε -Rules & Chomsky Normal Form

B8.1 Context-free Grammars and ε -Rules

B8.2 Chomsky Normal Form

B8.3 Summary

Content of the Course



B8.1 Context-free Grammars and ε -Rules

Repetition: Context-free Grammars

Definition (Context-free Grammar)

A **context-free grammar** is a 4-tuple $\langle V, \Sigma, P, S \rangle$ with

- 1 V finite set of variables,
- 2 Σ finite alphabet of terminal symbols (with $V \cap \Sigma = \emptyset$),
- 3 $P \subseteq (V \times (V \cup \Sigma)^+) \cup \{\langle S, \epsilon \rangle\}$ finite set of rules,
- 4 If $S \rightarrow \epsilon \in P$, then all other rules in $V \times ((V \setminus \{S\}) \cup \Sigma)^+$.
- 5 $S \in V$ start variable.

- ▶ All rules have a single variable on the left hand side.
- ▶ If the right hand side of a rule is empty (ϵ), then the left hand side is the start variable.
- ▶ If there is a rule with an empty right hand side, then the start variable does not occur on the right hand side of any rule.

With regular grammars, these restrictions could be lifted.

How about context-free grammars?

Context-free Grammars: Exercise

We have used the pumping lemma for regular languages to show that $L = \{a^n b^n \mid n \in \mathbb{N}_0\}$ is not regular.

Show that it is context-free by specifying a suitable grammar G with $\mathcal{L}(G) = L$.



Repetition: Context-free Grammars

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How about context-free grammars?

Reminder: Start Variable in Right-Hand Side of Rules

For every type-0 language L there is a grammar where the start variable does not occur on the right-hand side of any rule.

Theorem

For every grammar $G = \langle V, \Sigma, P, S \rangle$ there is a grammar $G' = \langle V', \Sigma, P', S \rangle$ with rules $P' \subseteq (V' \cup \Sigma)^+ \times (V' \setminus \{S\} \cup \Sigma)^*$ such that $\mathcal{L}(G) = \mathcal{L}(G')$.

In the proof we constructed a suitable grammar, where the rules in P' were not fundamentally different from the rules in P :

- ▶ for rules from $V \times (V \cup \Sigma)^+$, we only introduced additional rules from $V' \times (V' \cup \Sigma)^+$, and
- ▶ for rules from $V \times \epsilon$, we only introduced rules from $V' \times \epsilon$, where $V' = V \cup \{S'\}$ for some new variable $S' \notin V$.

ε -Rules

Theorem

For every grammar G with rules $P \subseteq V \times (V \cup \Sigma)^*$
there is a context-free grammar G' with $\mathcal{L}(G) = \mathcal{L}(G')$.

Proof.

Let $G = \langle V, \Sigma, P, S \rangle$ be a grammar with $P \subseteq V \times (V \cup \Sigma)^*$.

Let $G' = \langle V', \Sigma, P', S \rangle$ be a grammar with $\mathcal{L}(G) = \mathcal{L}(G')$ with
 $P' \subseteq V' \times ((V' \setminus S) \cup \Sigma)^*$.

Let $V_\varepsilon = \{A \in V' \mid A \Rightarrow_{G'}^* \varepsilon\}$. We can find this set V_ε by first
collecting all variables A with rule $A \rightarrow \varepsilon \in P'$ and then
successively adding additional variables B if there is a rule
 $B \rightarrow A_1 A_2 \dots A_k \in P'$ and the variables A_i are already in the set
for all $1 \leq i \leq k$

ε -Rules

Theorem

For every grammar G with rules $P \subseteq V \times (V \cup \Sigma)^*$
there is a context-free grammar G' with $\mathcal{L}(G) = \mathcal{L}(G')$.

Proof (continued).

Let P'' be the rule set that is constructed from P' by

- ▶ adding rules that obviate the need for $A \rightarrow \varepsilon$ rules:
for every existing rule $B \rightarrow w$ with $B \in V'$, $w \in (V' \cup \Sigma)^+$,
let I_ε be the set of positions where w contains a variable
 $A \in V_\varepsilon$. For every non-empty set $I' \subseteq I_\varepsilon$, add a new rule
 $B \rightarrow w'$, where w' is constructed from w by removing
the variables at all positions in I' .
- ▶ removing all rules of the form $A \rightarrow \varepsilon$ ($A \neq S$).

Then $G'' = \langle V', \Sigma, P'', S \rangle$ is context-free and $\mathcal{L}(G) = \mathcal{L}(G'')$. \square

Example

Consider $G = \langle \{X, Y, Z, S\}, \{a, b\}, R, S \rangle$ with rules:

$$S \rightarrow \varepsilon \mid XY$$

$$X \rightarrow aXYbX \mid YZ$$

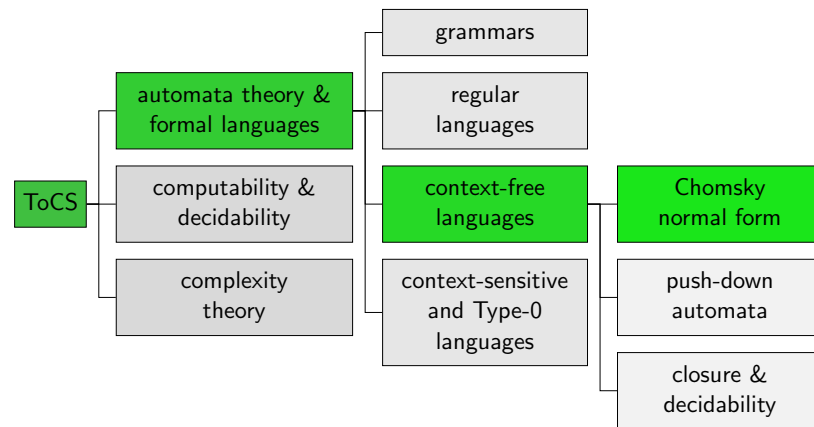
$$Y \rightarrow \varepsilon \mid b$$

$$Z \rightarrow \varepsilon \mid a$$

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B8.2 Chomsky Normal Form

Content of the Course



Chomsky Normal Form: Motivation

As in logical formulas (and other kinds of structured objects), **normal forms** for grammars are useful:

- ▶ they show which aspects are critical for defining grammars and which ones are just syntactic sugar
- ▶ they allow proofs and algorithms to be restricted to a limited set of grammars (inputs): those in normal form

Hence we now consider a **normal form** for context-free grammars.

Chomsky Normal Form: Definition

Definition (Chomsky Normal Form)

A context-free grammar G is in **Chomsky normal form (CNF)** if all rules have one of the following three forms:

- ▶ $A \rightarrow BC$ with variables A, B, C , or
- ▶ $A \rightarrow a$ with variable A , terminal symbol a , or
- ▶ $S \rightarrow \epsilon$ with start variable S .

in short:

rule set $P \subseteq (V \times (V'V' \cup \Sigma)) \cup \{(S, \epsilon)\}$ with $V' = V \setminus \{S\}$

German: Chomsky-Normalform

Chomsky Normal Form: Theorem

Theorem

For every context-free grammar G there is a context-free grammar G' in Chomsky normal form with $\mathcal{L}(G) = \mathcal{L}(G')$.

Proof.

The following algorithm converts the rule set of G into CNF:

Step 1: Eliminate rules of the form $A \rightarrow B$ with variables A, B .

If there are sets of variables $\{B_1, \dots, B_k\}$ with rules $B_1 \rightarrow B_2, B_2 \rightarrow B_3, \dots, B_{k-1} \rightarrow B_k, B_k \rightarrow B_1$, then replace these variables by a new variable B .

Define a strict total order $<$ on the variables such that $A \rightarrow B \in P$ implies that $A < B$. Iterate from the largest to the smallest variable A and eliminate all rules of the form $A \rightarrow B$ while adding rules $A \rightarrow w$ for every rule $B \rightarrow w$ with $w \in (V \cup \Sigma)^+$

Chomsky Normal Form: Theorem

Theorem

For every context-free grammar G there is a context-free grammar G' in Chomsky normal form with $\mathcal{L}(G) = \mathcal{L}(G')$.

Proof (continued).

Step 2: Eliminate rules with terminal symbols on the right-hand side that do not have the form $A \rightarrow a$.

For every terminal symbol $a \in \Sigma$ add a new variable A_a and the rule $A_a \rightarrow a$.

Replace all terminal symbols in all rules that do not have the form $A \rightarrow a$ with the corresponding newly added variables. ...

Chomsky Normal Form: Theorem

Theorem

For every context-free grammar G there is a context-free grammar G' in Chomsky normal form with $\mathcal{L}(G) = \mathcal{L}(G')$.

Proof (continued).

Step 3: Eliminate rules of the form $A \rightarrow B_1 B_2 \dots B_k$ with $k > 2$

For every rule of the form $A \rightarrow B_1 B_2 \dots B_k$ with $k > 2$, add new variables C_2, \dots, C_{k-1} and replace the rule with

$$\begin{aligned} A &\rightarrow B_1 C_2 \\ C_2 &\rightarrow B_2 C_3 \\ &\vdots \\ C_{k-1} &\rightarrow B_{k-1} B_k \end{aligned}$$

□

Example

Consider $G = (\{Y, Z, S\}, \{a, b\}, R, S)$ with rules:

$$S \rightarrow aZbY \mid Y \mid ab$$

$$Y \rightarrow Z \mid b$$

$$Z \rightarrow Y \mid bSa$$

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Chomsky Normal Form: Length of Derivations

Observation

Let G be a grammar in Chomsky normal form, and let $w \in \mathcal{L}(G)$ be a non-empty word generated by G .

Then all derivations of w have exactly $2|w| - 1$ derivation steps.

Why?

B8.3 Summary

Summary

- ▶ The restriction of ϵ -occurrences in rules is not necessary to characterize the set of context-free languages.
- ▶ Every context-free language has a grammar in **Chomsky normal form**. All rules have form
 - ▶ $A \rightarrow BC$ with variables A, B, C , or
 - ▶ $A \rightarrow a$ with variable A , terminal symbol a , or
 - ▶ $S \rightarrow \epsilon$ with start variable S .