

# Theory of Computer Science

## B1. Formal Languages & Grammars

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March 2/4, 2026

# Introduction

# Course Contents

Parts of the course:

A. background

▷ mathematical foundations and proof techniques

B. automata theory and formal languages

▷ What is a computation?

C. Turing computability ▷ What can be computed at all?

D. complexity theory ▷ What can be computed efficiently?

E. more computability theory ▷ Other models of computability

# A Controller for a Turnstile



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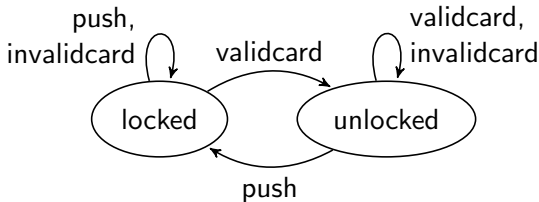
- simple access control
- card reader and push sensor
- card can either be valid or invalid

# A Controller for a Turnstile



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- simple access control
- card reader and push sensor
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# Turnstile Example: Decision Problem

## Definition (Decision Problem for Turnstile Example)

**Given:** Sequence of actions from set  
{push, validcard, invalidcard}

**Question:** If the turnstile was initially locked,  
is it unlocked after the given sequence of actions?  
That is, does the input sequence contain  
an action validcard such that afterwards  
there is never an occurrence of push?

# Decision Problems: Given-Question Form

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**Given:** possible input

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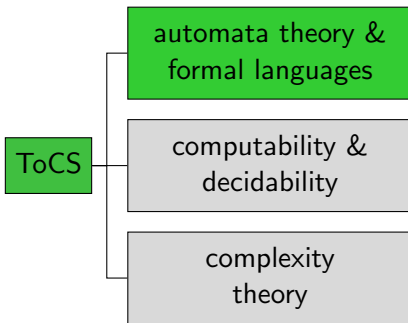
**Given:** possible input

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- often infinitely many instances (possible inputs).
- we want to characterize the set of all “Yes” instances
- **formal languages** are an alternative for representing such decision problems, using this set perspective instead of the given-question form.
- follow-up question: how can we characterize such a possibly infinite set with a final representation?

# Formal Languages

# Content of the Course



# Alphabets and Formal Languages

## Definition (Alphabets, Words and Formal Languages)

An **alphabet**  $\Sigma$  is a finite non-empty set of **symbols**.

## Example

$$\Sigma = \{a, b\}$$

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The **empty word** (the empty sequence of elements) is denoted by  $\varepsilon$ .

$\Sigma^*$  denotes the set of all words over  $\Sigma$ .

$\Sigma^+$  ( $= \Sigma^* \setminus \{\varepsilon\}$ ) denotes the set of all non-empty words over  $\Sigma$ .

## Example

$$\Sigma = \{a, b\}$$

$$\Sigma^* = \{\varepsilon, a, b, aa, ab, ba, bb, \dots\}$$

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We write  $|w|$  for the **length** of a word  $w$ .

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$$|aba| = 3, |b| = 1, |\varepsilon| = 0$$

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A **formal language** (over alphabet  $\Sigma$ ) is a subset of  $\Sigma^*$ .

## Example

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# Languages: Examples

Example (Languages over  $\Sigma = \{a, b\}$ )

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 $= \{\varepsilon, aab, aba, baa, \dots\}$
- $S_7 = \{w \in \Sigma^* \mid |w| = 3\}$   
 $= \{aaa, aab, aba, baa, bba, bab, abb, bbb\}$

# Languages: Turnstile Example

## Example

$$\Sigma = \{\text{push}, \text{validcard}, \text{invalidcard}\}$$
$$\mathcal{L}_{\text{turnstile}} = \{w \in \Sigma^* \mid \text{there is an occurrence of validcard in } w \\ \text{and after the last occurrence of validcard} \\ \text{there is no occurrence of push}\}$$

## Exercise (slido)

Consider  $\Sigma = \{\text{push}, \text{validcard}\}$ .

What is  $|\text{pushvalidcard}|$ ?



# Questions



Questions?

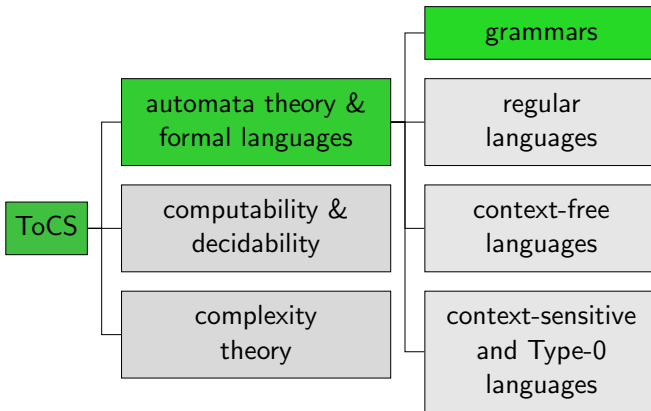
# Ways to Specify Formal Languages?

**Sought:** General concepts to define (often infinite) formal languages with finite descriptions.

- today: **grammars**
- later: automata, regular expressions, ...

# Grammars

# Content of the Course



# Grammar: Example

Variables  $V = \{S, X, Y\}$

Alphabet  $\Sigma = \{a, b, c\}$ .

Production rules:

$$S \rightarrow \varepsilon$$

$$S \rightarrow X$$

$$X \rightarrow aXYc$$

$$X \rightarrow abc$$

$$cY \rightarrow Yc$$

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You start from  $S$  and may in each step replace the left-hand side of a rule with the right-hand side of the same rule. This way, derive a word over  $\Sigma$ .

## Short-hand Notation for Rule Sets

We abbreviate several rules with the same left-hand side in a single line, using “|” for separating the right-hand sides.

For example, we write

$$X \rightarrow 0Y1 \mid XY$$

for:

$$X \rightarrow 0Y1 \text{ and}$$

$$X \rightarrow XY$$

## Exercise

Variables  $V = \{S, X, Y\}$

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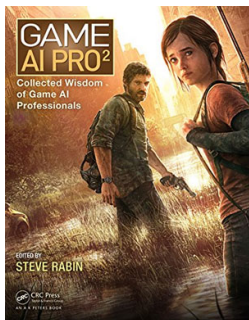
$$bY \rightarrow bb$$



Derive word  $aabbcc$  starting from  $S$ .

# Application: Content Generation in Games

- <http://www.gameaipro.com/>
- GameAIPro 2, chapter 40  
**Procedural Content Generation:  
An Overview** by Gillian Smith



# Questions



Questions?

# Grammars

## Definition (Grammars)

A **grammar** is a 4-tuple  $\langle V, \Sigma, R, S \rangle$  with:

- $V$  finite set of **variables** (**nonterminal symbols**)
- $\Sigma$  finite alphabet of **terminal symbols** with  $V \cap \Sigma = \emptyset$
- $R \subseteq (V \cup \Sigma)^* V (V \cup \Sigma)^* \times (V \cup \Sigma)^*$  finite set of **rules**
- $S \in V$  **start variable**

A rule is sometimes also called a **production** or a **production rule**.

## Rule Sets

What exactly does  $R \subseteq (V \cup \Sigma)^* V (V \cup \Sigma)^* \times (V \cup \Sigma)^*$  mean?

- $(V \cup \Sigma)^*$ : all words over  $(V \cup \Sigma)$

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- $\times$ : Cartesian product
- $(V \cup \Sigma)^* V (V \cup \Sigma)^* \times (V \cup \Sigma)^*$ : set of all pairs  $\langle x, y \rangle$ , where  $x$  is a word over  $(V \cup \Sigma)$  with at least one variable and  $y$  is a word over  $(V \cup \Sigma)$

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- Instead of  $\langle x, y \rangle$  we usually write rules in the form  $x \rightarrow y$ .

# Rules: Examples

## Example

Let  $\Sigma = \{a, b, c\}$  and  $V = \{X, Y, Z\}$ .

Some examples of rules in  $(V \cup \Sigma)^* V (V \cup \Sigma)^* \times (V \cup \Sigma)^*$ :

$$X \rightarrow XaY$$

$$Yb \rightarrow a$$

$$XY \rightarrow \varepsilon$$

$$XYZ \rightarrow abc$$

$$abXc \rightarrow XYZ$$

# Derivations

## Definition (Derivations)

Let  $\langle V, \Sigma, R, S \rangle$  be a grammar. A word  $v \in (V \cup \Sigma)^*$  can be **derived** from word  $u \in (V \cup \Sigma)^+$  (written as  $u \Rightarrow v$ ) if

- 1  $u = xyz$ ,  $v = xy'z$  with  $x, z \in (V \cup \Sigma)^*$  and
- 2 there is a rule  $y \rightarrow y' \in R$ .

We write:  $u \Rightarrow^* v$  if  $v$  can be derived from  $u$  in finitely many steps (i. e., by using  $n$  derivations for  $n \in \mathbb{N}_0$ ).

# Language Generated by a Grammar

## Definition (Languages)

The **language generated** by a grammar  $G = \langle V, \Sigma, P, S \rangle$

$$\mathcal{L}(G) = \{w \in \Sigma^* \mid S \Rightarrow^* w\}$$

is the set of all words from  $\Sigma^*$  that can be derived from  $S$  with finitely many rule applications.

# Grammars

Example (Languages over  $\Sigma = \{a, b\}$ )

- $L_1 = \{a, aa, aaa, aaaa, \dots\} = \{a\}^+$

Example grammars: blackboard

# Grammars

## Example (Languages over $\Sigma = \{a, b\}$ )

- $L_2 = \Sigma^*$

Example grammars: blackboard

# Grammars

## Example (Languages over $\Sigma = \{a, b\}$ )

- $L_3 = \{a^n b^n \mid n \geq 0\} = \{\varepsilon, ab, aabb, aaabbb, \dots\}$

Example grammars: blackboard

# Grammars

## Example (Languages over $\Sigma = \{a, b\}$ )

- $L_4 = \{\varepsilon\}$

Example grammars: blackboard

# Grammars

## Example (Languages over $\Sigma = \{a, b\}$ )

- $L_5 = \emptyset$

Example grammars: blackboard

# Grammars

## Example (Languages over $\Sigma = \{a, b\}$ )

- $L_6 = \{w \in \Sigma^* \mid w \text{ contains twice as many } a\text{'s as } b\text{'s}\}$   
 $= \{\varepsilon, aab, aba, baa, \dots\}$

Example grammars: blackboard

# Grammars

## Example (Turnstile)

$G = \langle \{S, U\}, \{\text{push, validcard, invalidcard}\}, R, S \rangle$   
with the following production rules in  $R$ :

$S \rightarrow \text{push } S$

$S \rightarrow \text{invalidcard } S$

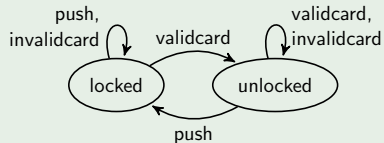
$S \rightarrow \text{validcard } U$

$U \rightarrow \text{invalidcard } U$

$U \rightarrow \text{validcard } U$

$U \rightarrow \epsilon$

$U \rightarrow \text{push } S$



$\mathcal{L}(G) = \mathcal{L}_{\text{turnstile}}$  from section “formal languages”

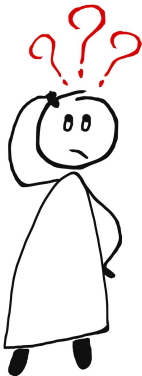
# Exercise

Specify a grammar that generates language

$$L = \{w \in \{a, b\}^* \mid |w| = 3\}.$$



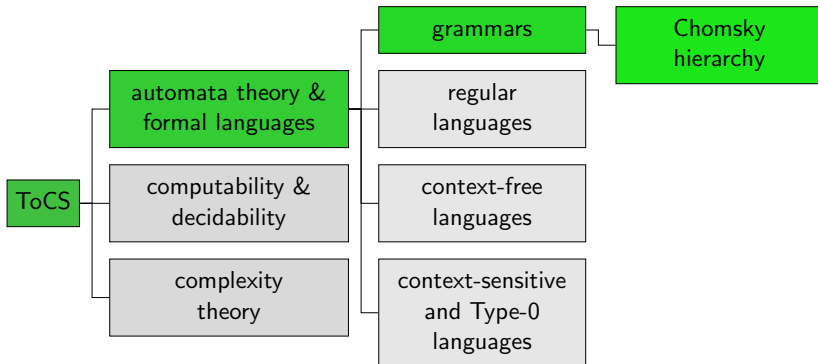
# Questions



Questions?

# Chomsky Hierarchy

# Content of the Course



# Noam Chomsky

- Avram Noam Chomsky (\*1928)
- "the father of modern linguistics"
- American linguist, philosopher, cognitive scientist, social critic, and political activist
- combined linguistics, cognitive science and computer science
- opponent of U.S. involvement in the Vietnam war
- there is a Wikipedia page solemnly on his political positions



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→ Organized grammars into the **Chomsky hierarchy**.

# Chomsky Hierarchy

## Definition (Chomsky Hierarchy)

- Every grammar is of **type 0** (all rules allowed).
- Grammar is of **type 1 (context-sensitive)**  
if all rules are of the form  $\alpha B \gamma \rightarrow \alpha \beta \gamma$   
with  $B \in V$  and  $\alpha, \gamma \in (V \cup \Sigma)^*$  and  $\beta \in (V \cup \Sigma)^+$
- Grammar is of **type 2 (context-free)**  
if all rules are of the form  $A \rightarrow w$ ,  
where  $A \in V$  and  $w \in (V \cup \Sigma)^+$ .
- Grammar is of **type 3 (regular)**  
if all rules are of the form  $A \rightarrow w$ ,  
where  $A \in V$  and  $w \in \Sigma \cup \Sigma V$ .

**special case:** rule  $S \rightarrow \varepsilon$  is always allowed if  $S$  is the start variable and never occurs on the right-hand side of any rule.

# Chomsky Hierarchy: Examples

Examples: blackboard

# Chomsky Hierarchy

## Definition (Type 0–3 Languages)

A language  $L \subseteq \Sigma^*$  is of type 0 (type 1, type 2, type 3) if there exists a type-0 (type-1, type-2, type-3) grammar  $G$  with  $\mathcal{L}(G) = L$ .

# Type $k$ Language: Example (slido)

## Example

Consider the language  $L$  generated by the grammar  $\langle \{F, A, N, C, D\}, \{a, b, c, \neg, \wedge, \vee, (, )\}, R, F \rangle$  with the following rules  $R$ :

$$F \rightarrow A$$

$$A \rightarrow a$$

$$N \rightarrow \neg F$$

$$F \rightarrow N$$

$$A \rightarrow b$$

$$C \rightarrow (F \wedge F)$$

$$F \rightarrow C$$

$$A \rightarrow c$$

$$D \rightarrow (F \vee F)$$

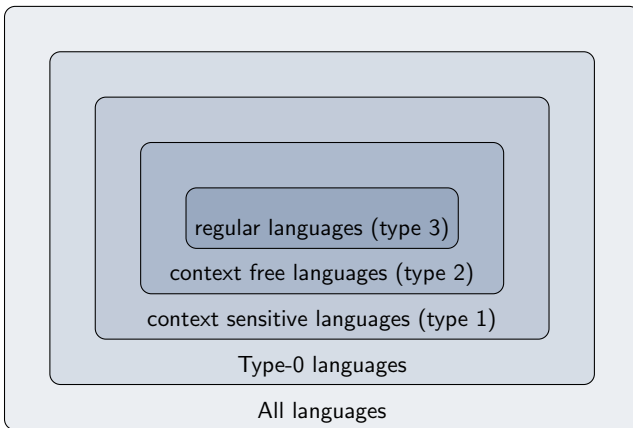
$$F \rightarrow D$$

### Questions:

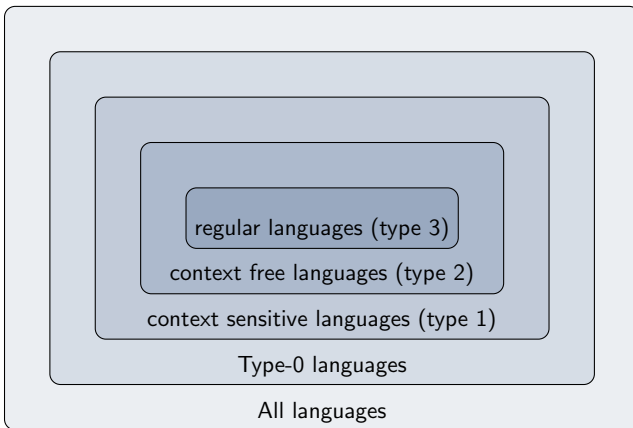
- Is  $L$  a type-0 language?
- Is  $L$  a type-1 language?
- Is  $L$  a type-2 language?
- Is  $L$  a type-3 language?



# Chomsky Hierarchy



# Chomsky Hierarchy



Note: Not all languages can be described by grammars. (Proof?)

# Questions



Questions?

# Summary

# Summary

- **Languages** are sets of symbol sequences.
- **Grammars** are one possible way to specify languages.
- Language **generated** by a grammar is the set of all words (of terminal symbols) **derivable** from the start symbol.
- **Chomsky hierarchy** distinguishes between languages at different levels of expressiveness.