

Theory of Computer Science

A3. Proof Techniques

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Proofs

What is a Proof?

A **mathematical proof** is

- a sequence of logical steps
- starting with one set of statements
- that comes to the conclusion that some statement must be true.

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What is a **statement**?

Mathematical Statements

Mathematical Statement

A **mathematical statement** is a declarative sentence that is either true or false (but not both).

Examples (some true, some false):

- Let $p \in \mathbb{N}_0$ be a prime number. Then p is odd.
- There exists an even prime number.
- The equation $a^k + b^k = c^k$ has infinitely many solutions with $a, b, c, k \in \mathbb{N}_1$ and $k \geq 2$.

German: Mathematische Aussage

Mathematical Statements: Quantification

Statements often use **quantification**.

- Universal quantification:

“For all x in set S it holds that ⟨sub-statement on x ⟩.”

This is **true** if the sub-statement is true for every x in S .

- Existential quantification:

“There is an x in set S such that ⟨sub-statement on x ⟩.”

This is **true** if there exists at least one x in S for which the sub-statement is true.

Examples (some true, some false):

- For all $x \in \mathbb{N}_1$ it holds that $x + 1$ is in \mathbb{N}_1 .
- For all $x \in \mathbb{N}_1$ it holds that $x - 1$ is in \mathbb{N}_1 .
- There is an $x \in \mathbb{N}_1$ such that $x = \sqrt{x}$.

Mathematical Statements: Preconditions and Conclusions

We can identify **preconditions** and **conclusions**.

“If ⟨preconditions⟩ then ⟨conclusions⟩.”

The statement is **true** if the conclusions are true whenever the preconditions are true.

Not every statement has preconditions. Preconditions are often used in universally quantified sub-statements.

Examples (some true, some false):

- If 4 is a prime number then $2 \cdot 3 = 4$.
- If n is a prime number with $n > 2$ then n is odd.
- For all $p \in \mathbb{N}_1$ it holds that if p is a prime number then p is odd.

Different Statements with the same Meaning

The following statements have the same meaning, we just move preconditions into the quantification, make some aspects implicit, and change the structure.

- For all $p \in \mathbb{N}_1$ it holds that if p is a prime number with $p > 2$ then p is odd.
- For all prime numbers p it holds that if $p > 2$ then p is odd.
- Let p be a natural number with $p > 2$.
Then p is odd if p is prime.
- If p is a prime number with $p > 2$ then p is odd.
- All prime numbers $p > 2$ are odd.

Different Statements with the same Meaning

The following statements have the same meaning, we just move preconditions into the quantification, make some aspects implicit, and change the structure.

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Then p is odd if p is prime.
- If p is a prime number with $p > 2$ then p is odd.
- All prime numbers $p > 2$ are odd.

A single mathematical statement can be expressed in different ways, as long as the meaning stays the same.

Like paraphrasing a sentence in everyday language.

On what Statements can we Build the Proof?

A mathematical proof is

- a sequence of logical steps
- **starting with one set of statements**
- that comes to the conclusion
that some statement must be true.

We can use:

- **axioms**: statements that are assumed to always be true in the current context
- **theorems** and **lemmas**: statements that were already proven
 - lemma: an intermediate tool
 - theorem: itself a relevant result
- **premises**: assumptions we make to see what consequences they have

German: Axiom, Theorem/Satz, Lemma, Prämisse/Annahme

What is a Logical Step?

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Each step **directly follows**

- from the axioms,
- premises,
- previously proven statements and
- the preconditions of the statement we want to prove.

What is a Logical Step?

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For a formal definition, we would need formal logics.

The Role of Definitions

Definition

A **set** is an unordered collection of distinct objects.

The objects in a set are called the **elements** of the set. A set is said to **contain** its elements.

We write $x \in S$ to indicate that x is an element of set S , and $x \notin S$ to indicate that S does not contain x .

The set that does not contain any objects is the **empty set** \emptyset .

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- A definition introduces an abbreviation.
- Whenever we say “set”, we could instead say “an unordered collection of distinct objects” and vice versa.
- Definitions can also introduce notation.

German: Definition

Disproofs

- A **disproof** (**refutation**) shows that a given mathematical statement is **false** by giving an example where the preconditions are true, but the conclusion is false.
- This requires deriving, in a sequence of proof steps, the opposite (negation) of the conclusion.

German: Widerlegung

Disproofs

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Example (False statement)

"If $p \in \mathbb{N}_0$ is a prime number then p is odd."

Refutation.

Consider natural number 2 as a counter example. It is prime because it has exactly 2 divisors, 1 and itself. It is not odd, because it is divisible by 2. □

German: Widerlegung

Exercise

You want to disprove the following statement with a counterexample:

If the sun is shining then all kids eat ice cream.

What properties must your counterexample have?

[Discuss with your neighbour; 2 minutes]



A Word on Style

A proof should help the reader to see why the result must be true.

- A proof should be easy to follow.
- Omit unnecessary information.
- Move self-contained parts into separate lemmas.
- In complicated proofs, reveal the overall structure in advance.
- Have a clear line of argument.

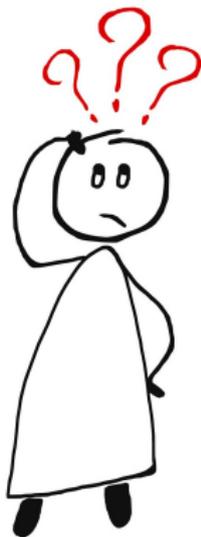
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- In complicated proofs, reveal the overall structure in advance.
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→ Writing a proof is like writing an essay.

Questions



Questions?

Proof Strategies

Proof Strategies

typical proof/disproof strategies:

- 1 “All $x \in S$ with the property P also have the property Q .”
“For all $x \in S$: if x has property P , then x has property Q .”
 - To prove, assume you are given an arbitrary $x \in S$ that has the property P .
Give a sequence of proof steps showing that x must have the property Q .
 - To disprove, find a **counterexample**, i. e., find an $x \in S$ that has property P but not Q and prove this.

Proof Strategies

typical proof/disproof strategies:

- ② “ A is a subset of B .”
 - To prove, assume you have an arbitrary element $x \in A$ and prove that $x \in B$.
 - To disprove, find an element in $x \in A \setminus B$ and prove that $x \in A \setminus B$.

Proof Strategies

typical proof/disproof strategies:

- ③ “For all $x \in S$: x has property P iff x has property Q .”
 (“iff”: “if and only if”)
 - To prove, separately prove “if P then Q ” and “if Q then P ”.
 - To disprove, disprove “if P then Q ” or disprove “if Q then P ”.

Proof Strategies

typical proof/disproof strategies:

- ④ “ $A = B$ ”, where A and B are sets.
 - To prove, separately prove “ $A \subseteq B$ ” and “ $B \subseteq A$ ”.
 - To disprove, disprove “ $A \subseteq B$ ” or disprove “ $B \subseteq A$ ”.

Proof Techniques

proof techniques we use in this course:

- direct proof
- indirect proof (proof by contradiction)

Direct Proof

Direct Proof

Direct Proof

Direct derivation of the statement by deducing or rewriting.

German: Direkter Beweis

Direct Proof: Example

Theorem (distributivity)

For all sets A, B, C : $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Direct Proof: Example

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For all sets A, B, C : $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof.

We first show that $x \in A \cap (B \cup C)$ implies
 $x \in (A \cap B) \cup (A \cap C)$ (\subseteq part):

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Let $x \in A \cap (B \cup C)$. Then by the definition of \cap it holds that
 $x \in A$ and $x \in B \cup C$.

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Let $x \in A \cap (B \cup C)$. Then by the definition of \cap it holds that
 $x \in A$ and $x \in B \cup C$.

We make a case distinction between $x \in B$ and $x \notin B$:

If $x \in B$ then, because $x \in A$ is true, $x \in A \cap B$ must be true.

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Let $x \in A \cap (B \cup C)$. Then by the definition of \cap it holds that $x \in A$ and $x \in B \cup C$.

We make a case distinction between $x \in B$ and $x \notin B$:

If $x \in B$ then, because $x \in A$ is true, $x \in A \cap B$ must be true.

Otherwise, because $x \in B \cup C$ we know that $x \in C$ and thus with $x \in A$, that $x \in A \cap C$.

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Theorem (distributivity)

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Proof.

We first show that $x \in A \cap (B \cup C)$ implies $x \in (A \cap B) \cup (A \cap C)$ (\subseteq part):

Let $x \in A \cap (B \cup C)$. Then by the definition of \cap it holds that $x \in A$ and $x \in B \cup C$.

We make a case distinction between $x \in B$ and $x \notin B$:

If $x \in B$ then, because $x \in A$ is true, $x \in A \cap B$ must be true.

Otherwise, because $x \in B \cup C$ we know that $x \in C$ and thus with $x \in A$, that $x \in A \cap C$.

In both cases $x \in A \cap B$ or $x \in A \cap C$,
and we conclude $x \in (A \cap B) \cup (A \cap C)$

Direct Proof: Example

Theorem (distributivity)

For all sets A, B, C : $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof (continued).

\supseteq part: we must show that $x \in (A \cap B) \cup (A \cap C)$ implies $x \in A \cap (B \cup C)$.

Let $x \in (A \cap B) \cup (A \cap C)$.

Direct Proof: Example

Theorem (distributivity)

For all sets A, B, C : $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof (continued).

\supseteq part: we must show that $x \in (A \cap B) \cup (A \cap C)$ implies $x \in A \cap (B \cup C)$.

Let $x \in (A \cap B) \cup (A \cap C)$.

We make a case distinction between $x \in A \cap B$ and $x \notin A \cap B$:

If $x \in A \cap B$ then $x \in A$ and $x \in B$.

The latter implies $x \in B \cup C$ and hence $x \in A \cap (B \cup C)$.

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Theorem (distributivity)

For all sets A, B, C : $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof (continued).

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Let $x \in (A \cap B) \cup (A \cap C)$.

We make a case distinction between $x \in A \cap B$ and $x \notin A \cap B$:

If $x \in A \cap B$ then $x \in A$ and $x \in B$.

The latter implies $x \in B \cup C$ and hence $x \in A \cap (B \cup C)$.

If $x \notin A \cap B$ we know $x \in A \cap C$ due to $x \in (A \cap B) \cup (A \cap C)$.

This (analogously) implies $x \in A$ and $x \in C$, and hence $x \in B \cup C$ and thus $x \in A \cap (B \cup C)$.

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Let $x \in (A \cap B) \cup (A \cap C)$.

We make a case distinction between $x \in A \cap B$ and $x \notin A \cap B$:

If $x \in A \cap B$ then $x \in A$ and $x \in B$.

The latter implies $x \in B \cup C$ and hence $x \in A \cap (B \cup C)$.

If $x \notin A \cap B$ we know $x \in A \cap C$ due to $x \in (A \cap B) \cup (A \cap C)$.

This (analogously) implies $x \in A$ and $x \in C$, and hence $x \in B \cup C$ and thus $x \in A \cap (B \cup C)$.

In both cases we conclude $x \in A \cap (B \cup C)$.

...

Direct Proof: Example

Theorem (distributivity)

For all sets A, B, C : $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof (continued).

We have shown that every element of $A \cap (B \cup C)$ is an element of $(A \cap B) \cup (A \cap C)$ and vice versa.
Thus, both sets are equal. □

Direct Proof: Example

Theorem (distributivity)

For all sets A, B, C : $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

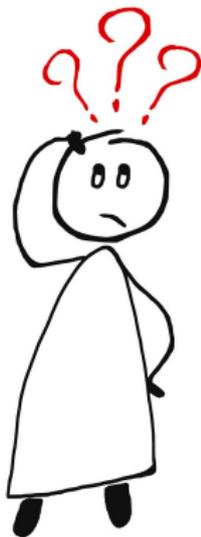
Proof.

Alternative:

$$\begin{aligned} A \cap (B \cup C) &= \{x \mid x \in A \text{ and } x \in B \cup C\} \\ &= \{x \mid x \in A \text{ and } (x \in B \text{ or } x \in C)\} \\ &= \{x \mid (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C)\} \\ &= \{x \mid x \in A \cap B \text{ or } x \in A \cap C\} \\ &= (A \cap B) \cup (A \cap C) \end{aligned}$$



Questions



Questions?

Indirect Proof

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Indirect Proof (Proof by Contradiction)

- Make an **assumption** that the statement is false.
- Use the assumption to derive a **contradiction**.
- This shows that the assumption must be false and hence the original statement must be true.

German: Indirekter Beweis, Beweis durch Widerspruch

Indirect Proof: Example 1

Theorem

Let A and B be sets. If $A \setminus B = \emptyset$ then $A \subseteq B$.

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Proof.

We prove the theorem by contradiction.

Assume that there are sets A and B with $A \setminus B = \emptyset$ and $A \not\subseteq B$.

Indirect Proof: Example 1

Theorem

Let A and B be sets. If $A \setminus B = \emptyset$ then $A \subseteq B$.

Proof.

We prove the theorem by contradiction.

Assume that there are sets A and B with $A \setminus B = \emptyset$ and $A \not\subseteq B$.

Let A and B be such sets.

Indirect Proof: Example 1

Theorem

Let A and B be sets. If $A \setminus B = \emptyset$ then $A \subseteq B$.

Proof.

We prove the theorem by contradiction.

Assume that there are sets A and B with $A \setminus B = \emptyset$ and $A \not\subseteq B$.

Let A and B be such sets.

Since $A \not\subseteq B$ there is some $x \in A$ such that $x \notin B$.

Indirect Proof: Example 1

Theorem

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Proof.

We prove the theorem by contradiction.

Assume that there are sets A and B with $A \setminus B = \emptyset$ and $A \not\subseteq B$.

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For this x it holds that $x \in A \setminus B$.

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Proof.

We prove the theorem by contradiction.

Assume that there are sets A and B with $A \setminus B = \emptyset$ and $A \not\subseteq B$.

Let A and B be such sets.

Since $A \not\subseteq B$ there is some $x \in A$ such that $x \notin B$.

For this x it holds that $x \in A \setminus B$.

This is a contradiction to $A \setminus B = \emptyset$.

We conclude that the assumption was false and thus the theorem is true. □

Indirect Proof: Example 2

Theorem

There are infinitely many prime numbers.

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Let $P = \{p_1, \dots, p_n\}$ be the set of all prime numbers.

Define $m = p_1 \cdot \dots \cdot p_n + 1$.



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Let $P = \{p_1, \dots, p_n\}$ be the set of all prime numbers.

Define $m = p_1 \cdot \dots \cdot p_n + 1$.

Since $m \geq 2$, it must have a prime factor.

Let p be such a prime factor.



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Since p is a prime number, p has to be in P .



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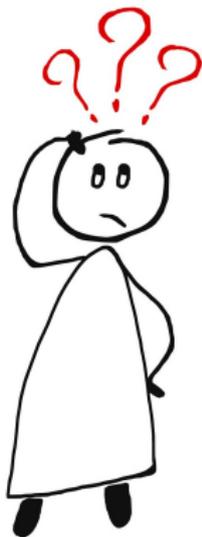
Since p is a prime number, p has to be in P .

The number m is not divisible without remainder by any of the numbers in P . Hence p is no factor of m .

↪ **Contradiction**

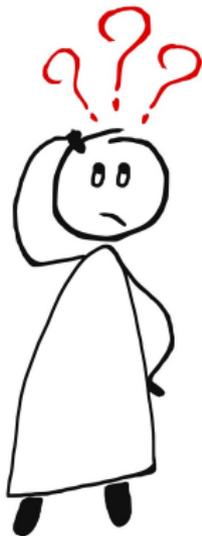


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Summary

Summary

- A **proof** is based on axioms and previously proven statements.
- Individual **proof steps** must be obvious derivations.
- **direct proof**: sequence of derivations or rewriting
- **indirect proof**: refute that the statement is false.