# Theory of Computer Science
## A2. Mathematical Foundations

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## A2.1 Sets, Tuples, Relations

## A2.2 Functions

## A2.3 Summary

# A2.1 Sets, Tuples, Relations

# Sets

- set: unordered collection of distinguishable objects;
  each object contained at most once
- notations:
    - explicit, listing all elements, e. g. $A = \{1, 2, 3\}$
    - implicit, specifying a property characterizing all elements,
      e. g. $A = \{x \mid x \in \mathbb{N} \text{ and } 1 \leq x \leq 3\}$
    - implicit, as a sequence with dots,
      e. g. $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- $e \in M$: $e$ is in set $M$ (an element of the set)
- $e \notin M$: $e$ is not in set $M$
- empty set $\emptyset = \{\}$
- cardinality $|M|$ of a finite set $M$: number of elements in $M$

German: Menge, Element, leere Menge, Mächtigkeit/Kardinalität

# Sets

- ▶ $A \subseteq B$: $A$ is a subset of $B$,
  i.e., every element of $A$ is an element of $B$
- ▶ $A \subset B$: $A$ is a strict subset of $B$,
  i.e., $A \subseteq B$ and $A \neq B$.
- ▶ power set $\mathcal{P}(M)$: set of all subsets of $M$
  e.g., $\mathcal{P}(\{a, b\}) =$
- ▶ Cardinality of power set of finite set $S$: $|\mathcal{P}(S)| =$

German: Teilmenge, echte Teilmenge, Potenzmenge

# Set Operations

▶ intersection $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$



▶ union $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$



▶ difference $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$



▶ complement $\overline{A} = B \setminus A$, where $A \subseteq B$ and
   $B$ is the set of all considered objects (in a given context)



German: Schnitt, Vereinigung, Differenz,
Komplement

# Tuples

- ▶ *k*-tuple: ordered sequence of $k$ objects
- ▶ written $(o_1, \ldots, o_k)$ or $\langle o_1, \ldots, o_k \rangle$
- ▶ unlike sets, order matters ($\langle 1, 2 \rangle \neq \langle 2, 1 \rangle$)
- ▶ objects may occur multiple times in a tuple
- ▶ objects contained in tuples are called components
- ▶ terminology:
  - ▶ $k = 2$: (ordered) pair
  - ▶ $k = 3$: triple
- ▶ if $k$ is clear from context (or does not matter), often just called tuple

German: *k*-Tupel, Komponente, Paar, Tripel

# Cartesian Product

- for sets $M_1, M_2, \ldots, M_n$, the Cartesian product $M_1 \times \cdots \times M_n$ is the set $M_1 \times \cdots \times M_n = \{\langle o_1, \ldots, o_n \rangle \mid o_1 \in M_1, \ldots, o_n \in M_n\}$.

- Example: $M_1 = \{a, b, c\}, M_2 = \{1, 2\}$, $M_1 \times M_2 = \{\langle a, 1 \rangle, \langle a, 2 \rangle, \langle b, 1 \rangle, \langle b, 2 \rangle, \langle c, 1 \rangle, \langle c, 2 \rangle\}$

- special case: $M^k = M \times \cdots \times M$ ($k$ times)

- example with $M = \{1, 2\}$: $M^2 = \{\langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 2, 1 \rangle, \langle 2, 2 \rangle\}$

German: kartesisches Produkt

# Relations

- an *n*-ary relation $R$ over the sets $M_1, \ldots, M_n$
  is a subset of their Cartesian product: $R \subseteq M_1 \times \cdots \times M_n$.
- example with $M = \{1, 2\}$:
  $R_{\leq} \subseteq M^2$ as $R_{\leq} = \{\langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 2, 2 \rangle\}$

German: (*n*-stellige) Relation

# A2.2 Functions

# Functions

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**Definition (Total Function)**

A (total) function $f : D \to C$ (with sets $D, C$)
maps every value of its domain $D$
to exactly one value of its codomain $C$.

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**Example**

- $square : \mathbb{Z} \to \mathbb{Z}$ with $square(x) = x^2$
- $add : \mathbb{N}_0^2 \to \mathbb{N}_0$ with $add(x, y) = x + y$
- $add_{\mathbb{R}} : \mathbb{R}^2 \to \mathbb{R}$ with $add_{\mathbb{R}}(x, y) = x + y$

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German: (totale) Funktion, Definitionsbereich, Wertebereich

## Functions: Example

### Example

Let $Q = \{q_0, q_1, q_2, q_{\text{accept}}, q_{\text{reject}}\}$ and $\Gamma = \{0, 1, \square\}$.

Define $\delta : (Q \setminus \{q_{\text{accept}}, q_{\text{reject}}\}) \times \Gamma \to Q \times \Gamma \times \{\mathsf{L}, \mathsf{R}\}$ by

| $\delta$ | 0 | 1 | $\square$ |
|---|---|---|---|
| $q_0$ | $\langle q_0, 0, \mathsf{R} \rangle$ | $\langle q_0, 1, \mathsf{R} \rangle$ | $\langle q_1, \square, \mathsf{L} \rangle$ |
| $q_1$ | $\langle q_2, 1, \mathsf{L} \rangle$ | $\langle q_1, 0, \mathsf{L} \rangle$ | $\langle q_{\text{reject}}, 1, \mathsf{L} \rangle$ |
| $q_2$ | $\langle q_2, 0, \mathsf{L} \rangle$ | $\langle q_2, 1, \mathsf{L} \rangle$ | $\langle q_{\text{accept}}, \square, \mathsf{R} \rangle$ |

Then, e. g., $\delta(q_0, 1) = \langle q_0, 1, \mathsf{R} \rangle$

# Partial Functions

> ### Definition (Partial Function)
> A partial function $f : X \rightarrow_p Y$ maps every value in $X$ to at most one value in $Y$.
>
> If $f$ does not map $x \in X$ to any value in $Y$, then $f$ is undefined for $x$.

> ### Example
> $f : \mathbb{N}_0 \times \mathbb{N}_0 \rightarrow_p \mathbb{N}_0$ with
>
> $$f(x,y) = \begin{cases} x - y & \text{if } y \leq x \\ \text{undefined} & \text{otherwise} \end{cases}$$

German: partielle Funktion

# A2.3 Summary

# Summary

▶ sets: unordered, contain every element at most once

▶ tuples: ordered, can contain the same object multiple times

▶ Cartesian product: $M_1 \times \cdots \times M_n$ set of all $n$-tuples where the $i$-th component is in $M_i$

▶ function $f : X \to Y$ maps every value in $X$ to exactly one value in $Y$

▶ partial function $g : X \to_p Y$ may be undefined for some values in $X$