

# Foundations of Artificial Intelligence

## G5. Board Games: Stochastic Games

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G5.1 Expected Value

G5.2 Stochastic Games

G5.3 Expectiminimax

G5.4 Summary

## Board Games: Overview

chapter overview:

- ▶ G1. Introduction and State of the Art
- ▶ G2. Formal Definition and Minimax Search
- ▶ G3. Evaluation Functions
- ▶ G4. Alpha-Beta Search
- ▶ **G5. Stochastic Games**
- ▶ G6. Monte-Carlo Tree Search Framework
- ▶ G7. Monte-Carlo Tree Search Variants

## G5.1 Expected Value

## Discrete Random Variable

- ▶ a **random event** (like the result of a die roll)
  - ▶ is described in terms of a **random variable**  $X$
  - ▶ with associated **domain**  $\text{dom}(X)$
  - ▶ and a **probability distribution** over the domain
- ▶ if the number of outcomes of a random event is **finite** (like here), the random variable is a **discrete random variable**
- ▶ and the probability distribution is given as a **probability**  $P(X = x)$  that the **outcome** is  $x \in \text{dom}(X)$

## Discrete Random Variable: Example

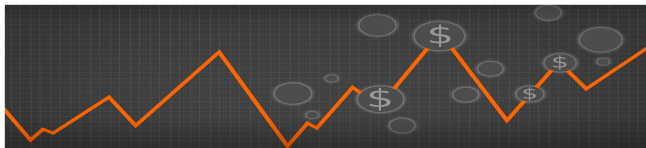


informal description:

- ▶ you plan to **invest** in **stocks** and can afford **one share**
- ▶ your analyst **expects** these **stock price changes**:

Bellman Inc.	Markov Tec.
+2 with 30%	+4 with 20%
+1 with 60%	+2 with 30%
±0 with 10%	-1 with 50%

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formal model:

- ▶ discrete random variables  $B$  and  $M$
- ▶  $\text{dom}(B) = \{2, 1, 0\}$
- ▶  $\text{dom}(M) = \{4, 2, -1\}$
- ▶  $P(B = 2) = 0.3$     $P(M = 4) = 0.2$
- ▶  $P(B = 1) = 0.6$     $P(M = 2) = 0.3$
- ▶  $P(B = 0) = 0.1$     $P(M = -1) = 0.5$

## Expected Value

- ▶ the **expected value**  $\mathbb{E}[X]$  of a random variable  $X$  is a **weighted average** of its outcomes
- ▶ it is computed as the **probability-weighted sum** of all outcomes  $x \in \text{dom}(X)$ , i.e.,

$$\mathbb{E}[X] = \sum_{x \in \text{dom}(X)} P(X = x) \cdot x$$

- ▶ in stochastic environments, it is **rational** to deal with uncertainty by **optimizing expected values**

## Expected Value: Example



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## Expected Value: Example



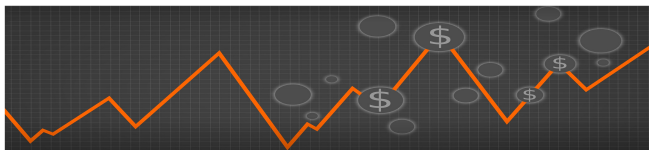
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## expected gain:

$$\begin{aligned}\mathbb{E}[B] &= P(B = 2) \cdot 2 + P(B = 1) \cdot 1 + P(B = 0) \cdot 0 \\ &= 0.3 \cdot 2 + 0.6 \cdot 1 + 0.1 \cdot 0 = 1.2 \\ \mathbb{E}[M] &= P(M = 4) \cdot 4 + P(M = 2) \cdot 2 + P(M = -1) \cdot (-1) \\ &= 0.2 \cdot 4 + 0.3 \cdot 2 + 0.5 \cdot (-1) = 0.9\end{aligned}$$

## Expected Value: Example



## formal model:

- ▶ discrete random variables  
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rational decision: buy Bellman Inc.

## G5.2 Stochastic Games

## Definition

### Definition (stochastic game)

A **stochastic game** is a

7-tuple  $\mathcal{S} = \langle S, A, T, s_1, S_G, utility, player \rangle$  with

- ▶ finite set of **positions**  $S$
- ▶ finite set of **moves**  $A$
- ▶ **transition function**  $T : S \times A \times S \mapsto [0, 1]$  that is **well-defined for**  $\langle s, a \rangle$  (see below)
- ▶ **initial position**  $s_1 \in S$
- ▶ set of **terminal positions**  $S_G \subseteq S$
- ▶ **utility function**  $utility : S_G \rightarrow \mathbb{R}$
- ▶ **player function**  $player : S \setminus S_G \rightarrow \{\text{MAX}, \text{MIN}\}$

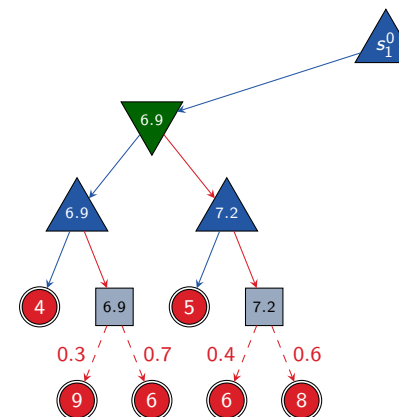
A transition function is **well-defined for**  $\langle s, a \rangle$  if  $\sum_{s' \in S} T(s, a, s') = 1$  (then  $a$  is **applicable** in  $s$ ) or  $\sum_{s' \in S} T(s, a, s') = 0$ .

## Example: Stochastic Inc-and-Square Game

- ▶ As an example, we consider a variant of the bounded inc-and-square game from Chapter G2.
- ▶ The **sqr** move now acts stochastically:
  - ▶ It **squares** the current value  $v$  (mod 10) with probability  $\frac{v}{10}$ .
  - ▶ Otherwise it **doubles** the current value  $v$  (mod 10) (with prob.  $1 - \frac{v}{10}$ ).
- ▶ We also reduce the maximum game length to 3 moves (counting both players) to make the example smaller.
- ▶ Everything else stays the same.

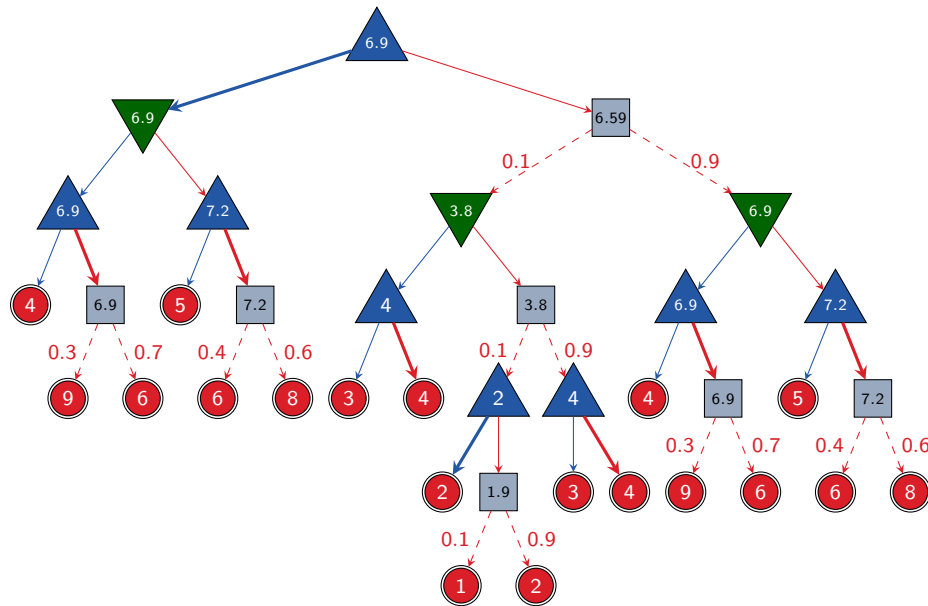
## G5.3 Expectiminimax

### Idea and Example



- ▶ **depth-first search** in game tree
- ▶ determine **utility value of terminal positions** with **utility function**
- ▶ compute **utility value of inner nodes** bottom-up through the tree:
  - ▶ MIN's turn: utility value is **minimum** of utility values of children
  - ▶ MAX's turn: utility value is **maximum** of utility values of children
  - ▶ chance: utility value is **expected value** of utility values of children
- ▶ **policy** for MAX: select action that leads to maximum utility value of children

## Idea and Example



## Discussion

- ▶ **expectiminimax** is the simplest (decent) search algorithm for stochastic games
- ▶ yields optimal policy (in the game-theoretic sense, i.e., under the assumption that the opponent plays perfectly)
- ▶ MAX obtains **at least** the utility value computed for the root **in expectation**, no matter how MIN plays
- ▶ if MIN plays perfectly, MAX obtains **exactly** the computed value **in expectation**

The same improvements as for minimax are possible (evaluation functions, alpha-beta search).

## G5.4 Summary

- ▶ **Stochastic games** are board games with an additional element of **chance**.
- ▶ **Expectiminimax** is a minimax variant for stochastic games with identical behavior in MAX and MIN nodes.
- ▶ In **chance nodes**, it propagates the **expected value** (probability-weighted sum) of all successors.
- ▶ Expectiminimax has **same guarantees** as minimax, but **in expectation**.

## Summary