Foundations of Artificial Intelligence

E3. Propositional Logic: Resolution

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Propositional Logic: Overview

Chapter overview: propositional logic

- E1. Syntax and Semantics
- E2. Equivalence, Normal Forms and Reasoning
- E3. Resolution
- E4. DPLL Algorithm
- E5. Local Search and Outlook

E3.1 Motivation

Reasoning and (Un-) Satisfiability

- last chapter: reduce reasoning to unsatisfiability
- also: interest in (un-) satisfiability in its own right

Can we test unsatisfiability in a more efficient way
than by computing the whole truth table?

- E3 & E4: complete algorithms for determining satisfiability/unsatisfiability of a formula
- E5: incomplete algorithms that find satisfying assignments

E3.2 Resolution

Sets of Clauses

in this chapter:

- prerequisite: formulas in conjunctive normal form
- clause represented as a set C of literals
- formula represented as a set Δ of clauses

Example

Let $\varphi = (P \lor Q) \land \neg P$.

- φ in conjunctive normal form
- φ consists of clauses $(P \lor Q)$ and $\neg P$
- representation of φ as set of sets of literals: $\{\{P, Q\}, \{\neg P\}\}$

Sets of Clauses (Corner Cases)

Distinguish $\bot$ (empty clause = empty set of literals)
vs. $\emptyset$ (empty set of clauses).

- $C = \bot$ $(= \emptyset)$ represents a disjunction over zero literals:

$$\bigvee_{L \in \emptyset} L = \bot$$

- $\Delta_1 = \{\bot\}$ represents a conjunction over one clause:

$$\bigwedge_{\varphi \in \{\bot\}} \varphi = \bot$$

- $\Delta_2 = \emptyset$ represents a conjunction over zero clauses:

$$\bigwedge_{\varphi \in \emptyset} \varphi = \top$$

Resolution: Idea

Resolution

- method to test CNF formula φ for unsatisfiability
- idea: derive new clauses from φ that logically follow from φ
- if empty clause $\bot$ can be derived $\rightsquigarrow$ φ unsatisfiable

German: Resolution

The Resolution Rule

$$\frac{C_1 \cup \{\ell\}, C_2 \cup \{\bar{\ell}\}}{C_1 \cup C_2}$$

- "From $C_1 \cup \{\ell\}$ and $C_2 \cup \{\bar{\ell}\}$, we can conclude $C_1 \cup C_2$."
- $C_1 \cup C_2$ is resolvent of parent clauses $C_1 \cup \{\ell\}$ and $C_2 \cup \{\bar{\ell}\}$.
- The literals ℓ and $\bar{\ell}$ are called resolution literals, the corresponding proposition is called resolution variable.
- resolvent follows logically from parent clauses (Why?)

German: Resolutionsregel, Resolvent, Elternklauseln, Resolutionsliterale, Resolutionsvariable

Example

- resolvent of $\{A, B, \neg C\}$ and $\{A, D, C\}$?
- resolvents of $\{\neg A, B, \neg C\}$ and $\{A, D, C\}$?

Resolution: Derivations

Definition (derivation)

Notation: $R(\Delta) = \Delta \cup \{C \mid C \text{ is resolvent of two clauses in } \Delta\}$

A clause D can be derived from Δ (in symbols $\Delta \vdash D$) if there is a sequence of clauses $C_1, \dots, C_n = D$ such that for all $i \in \{1, \dots, n\}$ we have $C_i \in R(\Delta \cup \{C_1, \dots, C_{i-1}\})$.

German: Ableitung, abgeleitet

Lemma (soundness of resolution)

If $\Delta \vdash D$, then $\Delta \models D$.

Does the converse direction hold as well (completeness)?

German: Korrektheit, Vollständigkeit

Resolution: Completeness?

The converse of the lemma does not hold in general.

example:

- $\{\{A, B\}, \{\neg B, C\}\} \models \{A, B, C\}$, but
- $\{\{A, B\}, \{\neg B, C\}\} \nvdash \{A, B, C\}$

but: converse holds for special case of empty clause $\bot$ (no proof)

Theorem (refutation-completeness of resolution)

Δ is unsatisfiable iff $\Delta \vdash \bot$

German: Widerlegungsvollständigkeit

consequences:

- Resolution is a complete proof method for testing unsatisfiability of CNF formulas.
- Resolution can be used for general reasoning by reducing to a test for unsatisfiability of CNF formulas.

Example

Let $\Phi = \{P \vee Q, \neg P\}$. Does $\Phi \models Q$ hold?

Solution

- test if $((P \vee Q) \wedge \neg P) \rightarrow Q$ is tautology
- equivalently: test if $((P \vee Q) \wedge \neg P) \wedge \neg Q$ is unsatisfiable
- resulting set of clauses: $\Phi' = \{\{P, Q\}, \{\neg P\}, \{\neg Q\}\}$
- resolving $\{P, Q\}$ with $\{\neg P\}$ yields $\{Q\}$
- resolving $\{Q\}$ with $\{\neg Q\}$ yields $\bot$
- observation: empty clause can be derived, hence Φ' unsatisfiable
- consequently $\Phi \models Q$

Resolution: Discussion

- Resolution is a complete proof method to test formulas for unsatisfiability.
- In the worst case, resolution proofs can take exponential time.
- In practice, a strategy which determines the next resolution step is needed.
- In the following chapter, we discuss the DPLL algorithm, which is a combination of backtracking and resolution.

E3.3 Summary

Summary

- resolution rule (for clauses in set notation): "from $C_1 \cup \{\ell\}$ and $C_2 \cup \{\bar{\ell}\}$, conclude $C_1 \cup C_2$."
- resolution: repeatedly apply the rule to see if a contradiction can be derived (empty clause $\bot$)
- is a refutation-complete proof method applicable to formulas in conjunctive normal form.

$\rightsquigarrow$ can be used to test if a set of clauses is unsatisfiable