

# Foundations of Artificial Intelligence

## E2. Propositional Logic: Equivalence, Normal Forms and Reasoning

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E2.1 Equivalence

E2.2 Normal Forms

E2.3 Reasoning

E2.4 Summary

# Propositional Logic: Overview

## Chapter overview: propositional logic

- ▶ E1. Syntax and Semantics
- ▶ E2. Equivalence, Normal Forms and Reasoning
- ▶ E3. Resolution
- ▶ E4. DPLL Algorithm
- ▶ E5. Local Search and Outlook

## E2.1 Equivalence

# Logical Equivalence

## Definition (logically equivalent)

Formulas  $\varphi$  and  $\psi$  are called **logically equivalent** ( $\varphi \equiv \psi$ ) if for all interpretations  $I$ :  $I \models \varphi$  iff  $I \models \psi$ .

**German:** logisch äquivalent

# Equivalences

## Logical Equivalences

Let  $\varphi$ ,  $\psi$ , and  $\eta$  be formulas.

- ▶  $(\varphi \wedge \psi) \equiv (\psi \wedge \varphi)$  and  $(\varphi \vee \psi) \equiv (\psi \vee \varphi)$  (commutativity)
- ▶  $((\varphi \wedge \psi) \wedge \eta) \equiv (\varphi \wedge (\psi \wedge \eta))$  and  
 $((\varphi \vee \psi) \vee \eta) \equiv (\varphi \vee (\psi \vee \eta))$  (associativity)
- ▶  $((\varphi \wedge \psi) \vee \eta) \equiv ((\varphi \vee \eta) \wedge (\psi \vee \eta))$  and  
 $((\varphi \vee \psi) \wedge \eta) \equiv ((\varphi \wedge \eta) \vee (\psi \wedge \eta))$  (distributivity)
- ▶  $\neg(\varphi \wedge \psi) \equiv (\neg\varphi \vee \neg\psi)$  and  
 $\neg(\varphi \vee \psi) \equiv (\neg\varphi \wedge \neg\psi)$  (De Morgan)
- ▶  $\neg\neg\varphi \equiv \varphi$  (double negation)
- ▶  $(\varphi \rightarrow \psi) \equiv (\neg\varphi \vee \psi)$  ( $(\rightarrow)$ -elimination)

Commutativity and associativity are often used implicitly

$\rightsquigarrow$  We write  $(X_1 \wedge X_2 \wedge X_3 \wedge X_4)$  instead of  $(X_1 \wedge (X_2 \wedge (X_3 \wedge X_4)))$

## E2.2 Normal Forms

# Normal Forms: Terminology

## Definition (literal)

If  $P \in \Sigma$ , then the formulas  $P$  and  $\neg P$  are called **literals**.

$P$  is called **positive literal**,  $\neg P$  is called **negative literal**.

The **complementary literal** to  $P$  is  $\neg P$  and vice versa.

For a literal  $\ell$ , the complementary literal to  $\ell$  is denoted with  $\bar{\ell}$ .

**German:** Literal, positives/negatives/komplementäres Literal

**Question:** What is the difference between  $\bar{\ell}$  and  $\neg\ell$ ?

# Normal Forms: Terminology

## Definition (clause)

A disjunction of 0 or more literals is called a **clause**.

The **empty clause** (with 0 literals) is  $\perp$ .

Clauses consisting of exactly one literal are called **unit clauses**.

**German:** Klausel, leere Klausel, Einheitsklausel

## Definition (monomial)

A conjunction of 0 or more literals is called a **monomial**.

**German:** Monom

# Normal Forms

## Definition (normal forms)

A formula  $\varphi$  is in **conjunctive normal form** (CNF, clause form) if  $\varphi$  is a conjunction of 0 or more clauses:

$$\varphi = \bigwedge_{i=1}^n \left( \bigvee_{j=1}^{m_i} \ell_{i,j} \right)$$

A formula  $\varphi$  is in **disjunctive normal form** (DNF) if  $\varphi$  is a disjunction of 0 or more monomials:

$$\varphi = \bigvee_{i=1}^n \left( \bigwedge_{j=1}^{m_i} \ell_{i,j} \right)$$

**German:** konjunktive Normalform, disjunktive Normalform

# Normal Forms

For every propositional formula, there exists a logically equivalent propositional formula in CNF and in DNF.

## Conversion to CNF with equivalences

- ① eliminate implications

$$(\varphi \rightarrow \psi) \equiv (\neg\varphi \vee \psi) \quad ((\rightarrow)\text{-elimination})$$

- ② move negations inside

$$\neg(\varphi \wedge \psi) \equiv (\neg\varphi \vee \neg\psi) \quad (\text{De Morgan})$$

$$\neg(\varphi \vee \psi) \equiv (\neg\varphi \wedge \neg\psi) \quad (\text{De Morgan})$$

$$\neg\neg\varphi \equiv \varphi \quad (\text{double negation})$$

- ③ distribute  $\vee$  over  $\wedge$

$$((\varphi \wedge \psi) \vee \eta) \equiv ((\varphi \vee \eta) \wedge (\psi \vee \eta)) \quad (\text{distributivity})$$

- ④ simplify constant subformulas ( $\top$ ,  $\perp$ )

There are formulas  $\varphi$  for which every logically equivalent formula in CNF and DNF is exponentially longer than  $\varphi$ .

## E2.3 Reasoning

# Reasoning: Intuition

## Reasoning: Intuition

- ▶ Generally, formulas only represent an incomplete description of the world.
- ▶ In many cases, we want to know if a formula **logically follows** from (a set of) other formulas.
- ▶ What does this mean?

## Reasoning: Intuition

▶ **example:**  $\varphi = (P \vee Q) \wedge (R \vee \neg P) \wedge S$

▶  $S$  holds in every model of  $\varphi$ .

What about  $P$ ,  $Q$  and  $R$ ?

↪ consider all models of  $\varphi$ :

▶  $I_1 = \{P \mapsto \mathbf{F}, Q \mapsto \mathbf{T}, R \mapsto \mathbf{F}, S \mapsto \mathbf{T}\}$

▶  $I_2 = \{P \mapsto \mathbf{F}, Q \mapsto \mathbf{T}, R \mapsto \mathbf{T}, S \mapsto \mathbf{T}\}$

▶  $I_3 = \{P \mapsto \mathbf{T}, Q \mapsto \mathbf{F}, R \mapsto \mathbf{T}, S \mapsto \mathbf{T}\}$

▶  $I_4 = \{P \mapsto \mathbf{T}, Q \mapsto \mathbf{T}, R \mapsto \mathbf{T}, S \mapsto \mathbf{T}\}$

### Observation

▶ In all models of  $\varphi$ , the formula  $Q \vee R$  holds as well.

▶ We say: “ $Q \vee R$  **logically follows** from  $\varphi$ .”

# Reasoning: Formally

## Definition (logical consequence)

Let  $\Phi$  be a set of formulas. A formula  $\psi$  **logically follows** from  $\Phi$  (in symbols:  $\Phi \models \psi$ ) if all models of  $\Phi$  are also models of  $\psi$ .

**German:** logische Konsequenz, folgt logisch

In other words: for each interpretation  $I$ ,  
if  $I \models \varphi$  for all  $\varphi \in \Phi$ , then also  $I \models \psi$ .

## Question

How can we automatically compute if  $\Phi \models \psi$ ?

- ▶ One possibility: Build a truth table. (How?)
- ▶ Are there “better” possibilities that (potentially) avoid generating the whole truth table?

# Reasoning: Deduction Theorem

## Proposition (deduction theorem)

Let  $\Phi$  be a finite set of formulas and let  $\psi$  be a formula. Then

$$\Phi \models \psi \quad \text{iff} \quad \left( \bigwedge_{\varphi \in \Phi} \varphi \right) \rightarrow \psi \text{ is a tautology.}$$

## German: Deduktionsatz

### Proof.

$$\Phi \models \psi$$

iff for each interpretation  $I$ : if  $I \models \varphi$  for all  $\varphi \in \Phi$ , then  $I \models \psi$

iff for each interpretation  $I$ : if  $I \models \bigwedge_{\varphi \in \Phi} \varphi$ , then  $I \models \psi$

iff for each interpretation  $I$ :  $I \not\models \bigwedge_{\varphi \in \Phi} \varphi$  or  $I \models \psi$

iff for each interpretation  $I$ :  $I \models \left( \bigwedge_{\varphi \in \Phi} \varphi \right) \rightarrow \psi$

iff  $\left( \bigwedge_{\varphi \in \Phi} \varphi \right) \rightarrow \psi$  is tautology



# Reasoning by Unsatisfiability Testing

## Consequence of Deduction Theorem

Reasoning can be reduced to testing unsatisfiability.

**Question:** Does  $\Phi \models \psi$  hold?

**Idea:**

- ▶ Let  $\chi = (\bigwedge_{\varphi \in \Phi} \varphi) \rightarrow \psi$ .
- ▶ We know that  $\Phi \models \psi$  iff  $\chi$  is a tautology.
- ▶ A formula is a tautology iff its negation is unsatisfiable.
- ▶ Hence,  $\Phi \models \psi$  iff  $\neg\chi$  is unsatisfiable.
- ▶ Use equivalences:
 
$$\begin{aligned} \neg\chi &= \neg((\bigwedge_{\varphi \in \Phi} \varphi) \rightarrow \psi) \equiv \neg(\neg(\bigwedge_{\varphi \in \Phi} \varphi) \vee \psi) \\ &\equiv (\neg\neg(\bigwedge_{\varphi \in \Phi} \varphi) \wedge \neg\psi) \equiv \bigwedge_{\varphi \in \Phi} \varphi \wedge \neg\psi \end{aligned}$$
- ▶ We have that  $\Phi \models \psi$  iff  $\bigwedge_{\varphi \in \Phi} \varphi \wedge \neg\psi$  is unsatisfiable.

# Algorithm for Reasoning

Question: Does  $\Phi \models \psi$  hold?

Algorithm (given an algorithm for testing unsatisfiability):

- 1 Let  $\eta = \bigwedge_{\varphi \in \Phi} \varphi \wedge \neg\psi$ .
- 2 Test if  $\eta$  is unsatisfiable.
- 3 If yes, return " $\Phi \models \psi$ ".
- 4 Otherwise, return " $\Phi \not\models \psi$ ".

Can we test unsatisfiability in a more efficient way than by computing the whole truth table?

$\rightsquigarrow$  next chapters

## E2.4 Summary

# Summary

- ▶ two formulas are **logically equivalent** if they have the same models
- ▶ different kinds of formulas:
  - ▶ **atomic formulas** and **literals**
  - ▶ **clauses** and **monomials**
  - ▶ **conjunctive normal form** (CNF) and **disjunctive normal form** (DNF)
- ▶ for every formula, there is a logically equivalent formula in CNF and a logically equivalent formula in DNF
- ▶ **Reasoning**: the formula  $\psi$  **follows from** the set of formulas  $\Phi$  if all models of  $\Phi$  are also models of  $\psi$ .
- ▶ Reasoning can be reduced to testing validity (with the **deduction theorem**).
- ▶ Testing validity can be reduced to testing unsatisfiability.