

Foundations of Artificial Intelligence

E1. Propositional Logic: Syntax and Semantics

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Propositional Logic: Overview

Chapter overview: propositional logic

- **E1. Syntax and Semantics**
- E2. Equivalence, Normal Forms and Reasoning
- E3. Resolution
- E4. DPLL Algorithm
- E5. Local Search and Outlook

Classification

classification:

Propositional Logic

environment:

- static vs. dynamic
- deterministic vs. nondeterministic vs. stochastic
- fully observable vs. partially observable
- discrete vs. continuous
- single-agent vs. multi-agent

problem solving method:

- problem-specific vs. **general** vs. learning

(applications also in more complex environments)

Motivation

Propositional Logic: Motivation

propositional logic

- modeling and representing problems and knowledge
- basis for **general** problem descriptions and solving strategies
 \rightsquigarrow automated planning (Part F)
- allows for automated **reasoning**

German: Aussagenlogik, automatisches Schliessen

Relationship to CSPs

- **previous part:** constraint satisfaction problems
- satisfiability problem in propositional logic can be viewed as **non-binary CSP** over $\{\mathbf{F}, \mathbf{T}\}$
- formula encodes constraints
- solution: satisfying assignment of values to variables
- backtracking with inference \approx DPLL ([Chapter E4](#))

Propositional Logic: Description of State Spaces

propositional variables for missionaries and cannibals problem:

two-missionaries-are-on-left-shore

one-cannibal-is-on-left-shore

boat-is-on-left-shore

...

- problem description for general problem solvers
- states represented as truth values of atomic **propositions**

German: Aussagenvariablen

Propositional Logic: Intuition

propositions: atomic statements over the world
that cannot be divided further

Propositions with **logical connectives** like
“and”, “or” and “not” form the propositional formulas.

German: logische Verknüpfungen

Syntax and Semantics

Today, we define **syntax** and **semantics** of propositional logic.
↔ **reminder** from Discrete Mathematics in Computer Science

syntax:

- defines which **symbols** are allowed in formulas
(,), \neg , \wedge , A , B , C , X , \heartsuit , \rightarrow , \nearrow , ...?
- ... and which **sequences** of these symbols are correct formulas
 $(A \wedge B)$, $((A) \wedge B)$, $\wedge)A(B, \dots?$

semantics:

- defines the **meaning** of formulas
- uses **interpretations** to describe a possible world
 $I = \{A \mapsto \mathbf{T}, B \mapsto \mathbf{F}\}$
- tells us which formulas are true in which worlds

Syntax

Alphabet of Propositions

- Logical formulas use an **alphabet Σ of propositions**, for example $\Sigma = \{P, Q, R, S\}$ or $\Sigma = \{X_1, X_2, X_3, \dots\}$.
- We do not mention the alphabet in the following.
- More formally, all definitions are parameterized by Σ .

German: Alphabet

Syntax

Definition (propositional formula)

- \top and \perp are formulas (**constant true/constant false**).
- Every proposition in Σ is a formula (**atomic formula**).
- If φ is a formula, then $\neg\varphi$ is a formula (**negation**).
- If φ and ψ are formulas, then so are
 - $(\varphi \wedge \psi)$ (**conjunction**)
 - $(\varphi \vee \psi)$ (**disjunction**)
 - $(\varphi \rightarrow \psi)$ (**implication**)

German: aussagenlogische Formel, konstant wahr/falsch,
atomare Formel, Konjunktion, Disjunktion, Implikation

Note: minor differences to Discrete Mathematics course

Abbreviating Notations: Omitting Parenthesis

may omit redundant parentheses:

- outer parentheses of formula:
 - $(P \wedge Q) \vee R$ instead of $((P \wedge Q) \vee R)$
- multiple conjunctions/disjunctions:
 - $P \wedge Q \wedge \neg R \wedge S$ instead of $((((P \wedge Q) \wedge \neg R) \wedge S)$
- implicit **binding strength**: $(\neg) > (\wedge) > (\vee) > (\rightarrow)$:
 - $P \vee Q \wedge R$ instead of $P \vee (Q \wedge R)$
 - use responsibly

Abbreviating Notations: Prefix Notation

prefix notations used like \sum for sums:

- $\bigvee_{i=1}^4 X_i$ instead of $(X_1 \vee X_2 \vee X_3 \vee X_4)$
- $\bigwedge_{i=1}^3 Y_i$ instead of $(Y_1 \wedge Y_2 \wedge Y_3)$

Semantics

Intuition for Semantics

A formula is **true** or **false**
depending on the **interpretation** of the propositions.

Semantics: Intuition

- A proposition P is either true or false.
The truth value of P is determined by an **interpretation**.
- The truth value of a formula follows from the truth values of the propositions.

Example

example interpretations for $\varphi = (P \vee Q) \wedge R$:

- If P and Q are false and R is true, then φ is false.
- If P is false and Q and R are true, then φ is true.

Interpretations

Definition (interpretation)

An **interpretation** I is a function $I : \Sigma \rightarrow \{\mathbf{T}, \mathbf{F}\}$.

Interpretations are sometimes called **truth assignments**.

German: Interpretation/Belegung/Wahrheitsbelegung

The Semantics of Formulas

When is a formula φ true under interpretation I ?
symbolically: When does $I \models \varphi$ hold?

Definition (Models and the \models Relation)

The relation “ \models ” is a relation between interpretations and formulas and is defined as follows:

- $I \models \top$ and $I \not\models \perp$
- $I \models P$ if $I(P) = \mathbf{T}$ for $P \in \Sigma$
- $I \models \neg\varphi$ if $I \not\models \varphi$
- $I \models (\varphi \wedge \psi)$ if $I \models \varphi$ and $I \models \psi$
- $I \models (\varphi \vee \psi)$ if $I \models \varphi$ or $I \models \psi$
- $I \models (\varphi \rightarrow \psi)$ if $I \not\models \varphi$ or $I \models \psi$

If $I \models \varphi$ ($I \not\models \varphi$), we say φ is **true (false)** under I .

Examples

Example (Interpretation I)

$$I = \{P \mapsto \mathbf{T}, Q \mapsto \mathbf{T}, R \mapsto \mathbf{F}, S \mapsto \mathbf{F}\}$$

Which formulas are true under I ?

- $\varphi_1 = \neg(P \wedge Q) \wedge (R \wedge \neg S)$. Does $I \models \varphi_1$ hold?
- $\varphi_2 = (P \wedge Q) \wedge \neg(R \wedge \neg S)$. Does $I \models \varphi_2$ hold?
- $\varphi_3 = (R \rightarrow P)$. Does $I \models \varphi_3$ hold?

Properties of Formulas

Models of Formulas and Sets of Formulas

Definition (model)

An interpretation I is called a **model** of φ if $I \models \varphi$.

Definition ($I \models \Phi$)

Let Φ be a set of propositional formulas.

We write $I \models \Phi$ if $I \models \varphi$ for all $\varphi \in \Phi$.

Such an interpretation I is called a **model** of Φ .

If I is a model of formula φ , we also say “ I satisfies φ ” or “ φ holds under I ” (similarly for sets of formulas Φ).

German: Modell, erfüllt, gilt unter

Satisfiable, Unsatisfiable, Falsifiable, Valid

Definition (satisfiable etc.)

A formula φ is called

- **satisfiable** if there exists a model for φ
- **unsatisfiable** if there exists no model for φ
- **valid** (= a **tautology**) if all interpretations are models of φ
- **falsifiable** if not all interpretations are models of φ

German: erfüllbar, unerfüllbar, allgemeingültig (gültig, Tautologie), falsifizierbar

Truth Tables

Truth Tables

How to determine automatically if a given formula is (un)satisfiable, valid, falsifiable?

↪ simple method: **truth tables**

example: Is $\varphi = ((P \vee H) \wedge \neg H) \rightarrow P$ valid?

P	H	$P \vee H$	$((P \vee H) \wedge \neg H)$	$((P \vee H) \wedge \neg H) \rightarrow P$
F	F	F	F	T
F	T	T	F	T
T	F	T	T	T
T	T	T	F	T

$I \models \varphi$ for all interpretations I , hence φ is valid.

- Is it satisfiable/unsatisfiable/falsifiable?

Terminology (Side Note)

What does “ φ is true” mean?

- not formally defined
- the statement misses an interpretation
 - could be meant as “in the obvious interpretation”
in some cases
 - or as “in all possible interpretations” (tautology)
- imprecise language \rightsquigarrow avoid

Summary

Summary

- **Propositional logic** forms the basis for a general representation of problems and knowledge.
- **Propositions** (atomic formulas) are statements over the world that cannot be divided further.
- **Propositional formulas** combine constant and atomic formulas with \neg , \wedge , \vee , \rightarrow to more complex statements.
- **Interpretations** determine which atomic formulas are true and which ones are false.
- Interpretations making a formula true are called **models**.
- important properties a formula may have:
satisfiable, unsatisfiable, valid, falsifiable