

# Foundations of Artificial Intelligence

## D1. Constraint Satisfaction Problems: Introduction and Examples

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# Constraint Satisfaction Problems: Overview

## Chapter overview: constraint satisfaction problems

- D1–D2. Introduction
  - D1. Introduction and Examples
  - D2. Constraint Networks
- D3–D6. Basic Algorithms
- D7–D8. Problem Structure

# Classification

classification:

## Constraint Satisfaction Problems

environment:

- static vs. dynamic
- deterministic vs. nondeterministic vs. stochastic
- fully observable vs. partially observable
- discrete vs. continuous
- single-agent vs. multi-agent

problem solving method:

- problem-specific vs. general vs. learning

Special case of a **pure search** combinatorial optimization problem

# Introduction

# Constraints

## What is a Constraint?

a condition that every solution to a problem must satisfy

**German:** Einschränkung, Nebenbedingung (math.)

**Examples:** where do constraints occur?

- **mathematics:** requirements on solutions of optimization problems (e.g., equations, inequalities)
- **software testing:** specification of invariants to check data consistency (e.g., assertions)
- **databases:** integrity constraints

# Constraint Satisfaction Problems: Informally

## Given:

- set of **variables** with corresponding domains
- set of **constraints** that the variables must satisfy
  - most commonly **binary**, i.e., every constraint refers to **two** variables

## Solution:

- **assignment** to the variables that satisfies all constraints

**German:** Variablen, Constraints, binär, Belegung

# Examples

# Examples

## Examples

- 8 queens problem
- Latin squares
- Sudoku
- graph coloring
- satisfiability in propositional logic

**German:** 8-Damen-Problem, lateinische Quadrate, Sudoku, Graphfärbung, Erfüllbarkeitsproblem der Aussagenlogik

more complex examples:

- systems of equations and inequalities
- database queries

# Example: 8 Queens Problem (Reminder)

(reminder from previous two chapters)

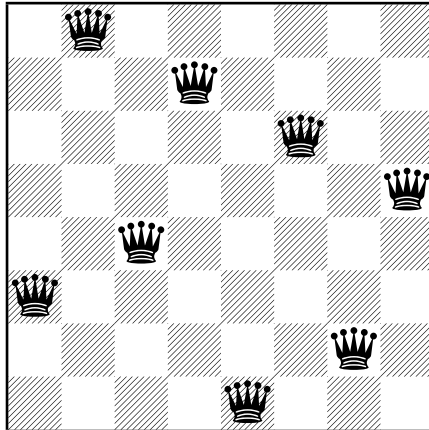
## 8 Queens Problem

How can we

- place **8 queens** on a chess board
  - such that **no two queens threaten each other?**
- 
- originally proposed in 1848
  - **variants**: board size; other pieces; higher dimension

There are **92 solutions**, or **12 solutions** if we do not count symmetric solutions (under rotation or reflection) as distinct.

# 8 Queens Problem: Example Solution



example solution for the 8 queens problem

# Example: Latin Squares

## Latin Squares

How can we

- build an  $n \times n$  matrix with  $n$  symbols
- such that every symbol occurs exactly once in every row and every column?

$$[1] \quad \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}$$

There exist 12 different Latin squares of size 3,  
576 of size 4, 161 280 of size 5, . . . ,  
5 524 751 496 156 892 842 531 225 600 of size 9.

# Example: Sudoku

## Sudoku

How can we

- completely fill an already partially filled  $9 \times 9$  matrix with numbers between 1–9
- such that each **row**, each **column**, and each of the nine  $3 \times 3$  **blocks** contains every number exactly once?

2	5			3		9		1
	1				4			
4		7				2		8
		5	2					
				9	8	1		
	4				3			
			3	6			7	2
	7							3
9		3				6		4

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6	1	9	8	2	4	3	5	7
4	3	7	9	1	5	2	6	8
3	9	5	2	7	1	4	8	6
7	6	2	4	9	8	1	3	5
8	4	1	6	5	3	7	2	9
1	8	4	3	6	9	5	7	2
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relationship to Latin squares?

# Sudoku: Trivia

- well-formed Sudokus have **exactly one** solution
- to achieve well-formedness,  $\geq 17$  cells must be filled already (McGuire et al., 2012)
- 6 670 903 752 021 072 936 960 solutions
- only 5 472 730 538 “non-symmetrical” solutions

# Example: Graph Coloring

## Graph Coloring

How can we

- color the vertices of a given graph using  $k$  colors
- such that two neighboring vertices never have the same color?

(The graph and  $k$  are problem parameters.)

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NP-complete problem

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- easy for  $k = 2$  (also for general graphs)

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Relationship to Sudoku?

# Four Color Problem

famous problem in mathematics: **Four Color Problem**

- Is it always possible to color a **planar** graph with 4 colors?
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- 1890 first proof that 5 colors suffice
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- 1890 first proof that 5 colors suffice
- several wrong proofs surviving for over 10 years
- solved by Appel and Haken in 1976: 4 colors suffice
- Appel and Haken reduced the problem to 1936 cases, which were then checked by computers
- first famous mathematical problem solved (partially) by computers
  - ↪ led to controversy: is this a mathematical proof?

Numberphile video:

<https://www.youtube.com/watch?v=NgBK43jB4rQ>

# Satisfiability in Propositional Logic

## Satisfiability in Propositional Logic

How can we

- assign **truth values** (true/false) to a set of propositional variables
- such that a given set of **clauses** (formulas of the form  $X \vee \neg Y \vee Z$ ) is satisfied (true)?

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remarks:

- NP-complete (Cook 1971; Levin 1973)
- requiring clause form (instead of arbitrary propositional formulas) is no restriction
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relationship to previous problems (e.g., Sudoku)?

# Practical Applications

- There are **thousands** of practical applications of constraint satisfaction problems.
- This statement is true already for the satisfiability problem of propositional logic.

some examples:

- verification of hardware and software
- timetabling (e.g., generating time schedules, room assignments for university courses)
- assignment of frequency spectra (e.g., broadcasting, mobile phones)

# Running Example

## Small Math Puzzle (informal description)

- assign a value from  $\{1, 2, 3, 4\}$  to the variables  $w$  and  $y$
- and from  $\{1, 2, 3\}$  to  $x$  and  $z$
- such that
  - $w = 2x$ ,
  - $w < z$  and
  - $y > z$ .

We will use this example to explain definitions and algorithms in the next chapters.

# Summary

# Summary

- **constraint satisfaction:**
  - find **assignment** for a set of **variables**
  - with given **variable domains**
  - that satisfies a given set of **constraints**.
- **examples:**
  - 8 queens problem
  - Latin squares
  - Sudoku
  - graph coloring
  - satisfiability in propositional logic
  - many practical applications