

Foundations of Artificial Intelligence

B15. State-Space Search: Properties of A^* , Part II

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State-Space Search: Overview

Chapter overview: state-space search

- B1–B3. Foundations
- B4–B8. Basic Algorithms
- B9–B15. Heuristic Algorithms
 - B9. Heuristics
 - B10. Analysis of Heuristics
 - B11. Best-first Graph Search
 - B12. Greedy Best-first Search, A^* , Weighted A^*
 - B13. IDA^{*}
 - B14. Properties of A^* , Part I
 - B15. Properties of A^* , Part II

Introduction

Optimality of A^* without Reopening

We now study A^* without reopening.

- For A^* without reopening, admissibility and consistency together guarantee optimality.
- We prove this on the following slides, again beginning with a basic lemma.
- Either of the two properties on its own would **not** be sufficient for optimality. (How would one prove this?)

Reminder: A* without Reopening

reminder from Chapter B11/B12: A* without reopening

A* without Reopening

```
open := new MinHeap ordered by  $\langle f, h \rangle$ 
if  $h(\text{init}()) < \infty$ :
    open.insert(make_root_node())
closed := new HashSet
while not open.is_empty():
    n := open.pop_min()
    if  $n.\text{state} \notin \text{closed}$ :
        closed.insert(n)
        if is_goal(n.state):
            return extract_path(n)
        for each  $\langle a, s' \rangle \in \text{succ}(n.\text{state})$ :
            if  $h(s') < \infty$ :
                n' := make_node(n, a, s')
                open.insert(n')
```

return unsolvable

Monotonicity Lemma

A*: Monotonicity Lemma (1)

Lemma (monotonicity of A* with consistent heuristics)

Consider A* with a *consistent* heuristic.

Then:

- 1 If n' is a child node of n , then $f(n') \geq f(n)$.
- 2 On all paths generated by A*, f values are non-decreasing.
- 3 The sequence of f values of the nodes expanded by A* is non-decreasing.

German: Monotonielemma

A^* : Monotonicity Lemma (2)

Proof.

on 1.:

Let n' be a child node of n via action a .

Let $s = n.\text{state}$, $s' = n'.\text{state}$.

A*: Monotonicity Lemma (2)

Proof.

on 1.:

Let n' be a child node of n via action a .

Let $s = n.state$, $s' = n'.state$.

- by definition of f : $f(n) = g(n) + h(s)$, $f(n') = g(n') + h(s')$

A^* : Monotonicity Lemma (2)

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- by definition of g : $g(n') = g(n) + cost(a)$

A*: Monotonicity Lemma (2)

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- by consistency of h : $h(s) \leq cost(a) + h(s')$

A^* : Monotonicity Lemma (2)

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$$\rightsquigarrow f(n) = g(n) + h(s) \leq g(n) + cost(a) + h(s') \\ = g(n') + h(s') = f(n')$$

A^* : Monotonicity Lemma (2)

Proof.

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Let n' be a child node of n via action a .

Let $s = n.state$, $s' = n'.state$.

- by definition of f : $f(n) = g(n) + h(s)$, $f(n') = g(n') + h(s')$
- by definition of g : $g(n') = g(n) + cost(a)$
- by consistency of h : $h(s) \leq cost(a) + h(s')$

$$\rightsquigarrow f(n) = g(n) + h(s) \leq g(n) + cost(a) + h(s') \\ = g(n') + h(s') = f(n')$$

on 2.: follows directly from 1.

...

A^* : Monotonicity Lemma (3)

Proof (continued).

on 3:

- Let f_b be the minimal f value in *open* at the beginning of a **while** loop iteration in A^* .
Let n be the removed node with $f(n) = f_b$.

A^* : Monotonicity Lemma (3)

Proof (continued).

on 3:

- Let f_b be the minimal f value in *open* at the beginning of a **while** loop iteration in A^* .
Let n be the removed node with $f(n) = f_b$.
- to show: at the end of the iteration the minimal f value in *open* is at least f_b .

A*: Monotonicity Lemma (3)

Proof (continued).

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- to show: at the end of the iteration the minimal f value in *open* is at least f_b .
- We must consider the operations modifying *open*: *open.pop_min* and *open.insert*.

A*: Monotonicity Lemma (3)

Proof (continued).

on 3:

- Let f_b be the minimal f value in *open* at the beginning of a **while** loop iteration in A*.
Let n be the removed node with $f(n) = f_b$.
- to show: at the end of the iteration the minimal f value in *open* is at least f_b .
- We must consider the operations modifying *open*: *open.pop_min* and *open.insert*.
- *open.pop_min* can never decrease the minimal f value in *open* (only potentially increase it).

A*: Monotonicity Lemma (3)

Proof (continued).

on 3:

- Let f_b be the minimal f value in *open* at the beginning of a **while** loop iteration in A*.
Let n be the removed node with $f(n) = f_b$.
- to show: at the end of the iteration the minimal f value in *open* is at least f_b .
- We must consider the operations modifying *open*: *open.pop_min* and *open.insert*.
- *open.pop_min* can never decrease the minimal f value in *open* (only potentially increase it).
- The nodes n' added with *open.insert* are children of n and hence satisfy $f(n') \geq f(n) = f_b$ according to part 1.



Optimality of A^* without Reopening

Optimality of A^* without Reopening

Theorem (optimality of A^* without reopening)

A^* *without reopening* is optimal when using an *admissible* and *consistent* heuristic.

Proof.

From the monotonicity lemma, the sequence of f values of nodes removed from the open list is non-decreasing.

- ↪ If multiple nodes with the same state s are removed from the open list, then their g values are non-decreasing.
- ↪ If we allowed reopening, it would never happen.
- ↪ With consistent heuristics, A^* without reopening behaves the same way as A^* with reopening.

The result follows because A^* with reopening and admissible heuristics is optimal.



Time Complexity of A^*

Time Complexity of A* (1)

What is the time complexity of A*?

- depends strongly on the quality of the heuristic
- an extreme case: $h = 0$ for all states
 - ↪ A* identical to uniform cost search
- another extreme case: $h = h^*$ and $cost(a) > 0$ for all actions a
 - ↪ A* only expands nodes along an optimal solution
 - ↪ $O(l^*)$ expanded nodes, $O(l^* b)$ generated nodes, where
 - l^* : length of the found optimal solution
 - b : branching factor

Time Complexity of A* (2)

more precise analysis:

- dependency of the runtime of A* on **heuristic error**

example:

- unit cost problems with
- **constant branching factor** and
- **constant absolute error**: $|h^*(s) - h(s)| \leq c$ for all $s \in S$

time complexity:

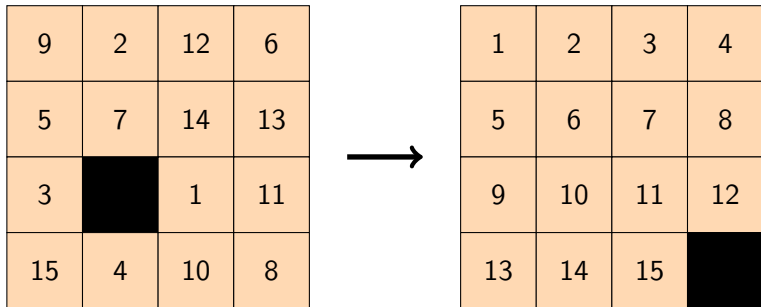
- **if state space is a tree**: time complexity of A* grows linearly in solution length (Pohl 1969; Gaschnig 1977)
- **general search spaces**: runtime of A* grows exponentially in solution length (Helmert & Röger 2008)

Overhead of Reopening

How does reopening affect runtime?

- For most practical state spaces and inconsistent admissible heuristics, the number of reopened nodes is **negligible**.
- **exceptions** exist:
Martelli (1977) constructed state spaces with n states where **exponentially** many (in n) node reopenings occur in A^* .
(\rightsquigarrow exponentially worse than uniform cost search)

Practical Evaluation of A^* (1)



h_1 : number of tiles in wrong cell (**misplaced tiles**)

h_2 : sum of distances of tiles to their goal cell (**Manhattan distance**)

Practical Evaluation of A* (2)

- experiments with random initial states, generated by **random walk** from goal state
- entries show **median** of number of **generated nodes** for 101 random walks of the same length N

	generated nodes		
N	BFS-Graph	A* with h_1	A* with h_2
10	63	15	15
20	1,052	28	27
30	7,546	77	42
40	72,768	227	64
50	359,298	422	83
60	> 1,000,000	7,100	307
70	> 1,000,000	12,769	377
80	> 1,000,000	62,583	849
90	> 1,000,000	162,035	1,522
100	> 1,000,000	690,497	4,964

Summary

Summary

- A* without reopening using an admissible and consistent heuristic is optimal
- key property **monotonicity lemma** (with consistent heuristics):
 - f values never decrease along paths considered by A*
 - sequence of f values of expanded nodes is non-decreasing
- time complexity depends on heuristic and shape of state space
 - precise details complex and depend on many aspects
 - reopening increases runtime exponentially in degenerate cases, but usually negligible overhead
 - small improvements in heuristic values often lead to exponential improvements in runtime