

# Foundations of Artificial Intelligence

## B14. State-Space Search: Properties of $A^*$ , Part I

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# State-Space Search: Overview

## Chapter overview: state-space search

- B1–B3. Foundations
- B4–B8. Basic Algorithms
- B9–B15. Heuristic Algorithms
  - B9. Heuristics
  - B10. Analysis of Heuristics
  - B11. Best-first Graph Search
  - B12. Greedy Best-first Search,  $A^*$ , Weighted  $A^*$
  - B13. IDA\*
  - B14. Properties of  $A^*$ , Part I
  - B15. Properties of  $A^*$ , Part II

# Introduction

# Optimality of $A^*$

- advantage of  $A^*$  over greedy search:  
    **optimal** for heuristics with suitable properties
- **very important result!**

↪ next chapters: a closer look at  $A^*$

- $A^*$  with reopening ↪ this chapter
- $A^*$  without reopening ↪ next chapter

# Optimality of $A^*$ with Reopening

In this chapter, we prove that  $A^*$  with reopening is optimal when using admissible heuristics.

For this purpose, we

- give some basic definitions
- prove two lemmas regarding the behaviour of  $A^*$
- use these to prove the main result

# Reminder: A\* with Reopening

reminder from Chapter B11/B12: A\* with reopening

## A\* with Reopening

```
open := new MinHeap ordered by  $\langle f, h \rangle$ 
if  $h(\text{init}()) < \infty$ :
    open.insert(make_root_node())
distances := new HashMap
while not open.is_empty():
    n := open.pop_min()
    if distances.lookup(n.state) = none or  $g(n) < \text{distances}[n.state]$ :
        distances[n.state] :=  $g(n)$ 
        if is_goal(n.state):
            return extract_path(n)
        for each  $\langle a, s' \rangle \in \text{succ}(n.state)$ :
            if  $h(s') < \infty$ :
                n' := make_node(n, a, s')
                open.insert(n')
return unsolvable
```

# Solvable States

## Definition (solvable)

A state  $s$  of a state space is called **solvable** if  $h^*(s) < \infty$ .

German: lösbar

# Optimal Paths to States

## Definition ( $g^*$ )

Let  $s$  be a state of a state space with initial state  $s_1$ .

We write  $g^*(s)$  for the cost of an optimal (cheapest) path from  $s_1$  to  $s$  ( $\infty$  if  $s$  is unreachable).

## Remarks:

- $g$  is defined for nodes,  $g^*$  for states (Why?)
- $g^*(n.state) \leq g(n)$  for all nodes  $n$  generated by a search algorithm (Why?)

# Settled States in A\*

## Definition (settled)

A state  $s$  is called **settled** at a given point during the execution of A\* (with or without reopening) if  $s$  is included in  $distances$  and  $distances[s] = g^*(s)$ .

German: erledigt

# Optimal Continuation Lemma

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We now show the first important result for  $A^*$  with reopening:

## Lemma (optimal continuation lemma)

Consider  $A^*$  with reopening using a *safe* heuristic at the beginning of any iteration of the **while** loop.

If

- state  $s$  is settled,
- state  $s'$  is a solvable successor of  $s$ , and
- an optimal path from  $s_1$  to  $s'$  of the form  $\langle s_1, \dots, s, s' \rangle$  exists,

then

- $s'$  is settled or
- open contains a node  $n'$  with  $n'.state = s'$  and  $g(n') = g^*(s')$ .

German: Optimale-Fortsetzungs-Lemma

# Optimal Continuation Lemma: Intuition

(Proof follows on the next slides.)

Intuitively, the lemma states:

*If no optimal path to a given state has been found yet, open must contain a “good” node that contributes to finding an optimal path to that state.*

(This potentially requires multiple applications of the lemma along an optimal path to the state.)

# Optimal Continuation Lemma: Proof (1)

## Proof.

Consider states  $s$  and  $s'$  with the given properties at the start of some iteration (“iteration  $A$ ”) of  $A^*$ .

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Because  $s$  is settled, an earlier iteration (“iteration B”) set  $distances[s] := g^*(s)$ .

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Thus iteration B removed a node  $n$  with  $n.state = s$  and  $g(n) = g^*(s)$  from *open*.

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Because  $s$  is settled, an earlier iteration (“iteration B”) set  $distances[s] := g^*(s)$ .

Thus iteration B removed a node  $n$  with  $n.state = s$  and  $g(n) = g^*(s)$  from *open*.

$A^*$  did not terminate in iteration B.  
(Otherwise iteration A would not exist.)

Hence  $n$  was expanded in iteration B.

...

# Optimal Continuation Lemma: Proof (2)

## Proof (continued).

This expansion considered the successor  $s'$  of  $s$ .

Because  $s'$  is solvable, we have  $h^*(s') < \infty$ .

Because  $h$  is safe, this implies  $h(s') < \infty$ .

Hence a successor node  $n'$  was generated for  $s'$ .

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This node  $n'$  satisfies the consequence of the lemma.

Hence the criteria of the lemma were satisfied for  $s$  and  $s'$  after iteration B.

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Hence the criteria of the lemma were satisfied for  $s$  and  $s'$  after iteration B.

To complete the proof, we show: if the consequence of the lemma is satisfied at the beginning of an iteration, it is also satisfied at the beginning of the next iteration. ...

# Optimal Continuation Lemma: Proof (3)

## Proof (continued).

- If  $s'$  is settled at the beginning of an iteration, it remains settled until termination.

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## Proof (continued).

- If  $s'$  is settled at the beginning of an iteration, it remains settled until termination.
- If  $s'$  is not yet settled and  $open$  contains a node  $n'$  with  $n'.state = s'$  and  $g(n') = g^*(s')$  at the beginning of an iteration, then either the node remains in  $open$  during the iteration, or  $n'$  is removed during the iteration and  $s'$  becomes settled.



# *f*-Bound Lemma

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We need a second lemma:

## Lemma (*f*-bound lemma)

Consider  $A^*$  with reopening and an *admissible* heuristic applied to a *solvable* state space with optimal solution cost  $c^*$ .

Then open contains a node  $n$  with  $f(n) \leq c^*$  at the beginning of each iteration of the **while** loop.

German: *f*-Schranken-Lemma

# *f*-Bound Lemma: Proof (1)

## Proof.

Consider the situation at the beginning of any iteration of the **while** loop.

Let  $\langle s_0, \dots, s_n \rangle$  with  $s_0 := s_1$  be an optimal solution.  
(Here we use that the state space is solvable.)

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Consider the situation at the beginning of any iteration of the **while** loop.

Let  $\langle s_0, \dots, s_n \rangle$  with  $s_0 := s_1$  be an optimal solution.  
(Here we use that the state space is solvable.)

Let  $s_i$  be the first state in the sequence that is not settled.

(Not all states in the sequence can be settled:  
 $s_n$  is a goal state, and when a goal state is inserted into *distances*,  $A^*$  terminates.)

...

# *f*-Bound Lemma: Proof (2)

Proof (continued).

Case 1:  $i = 0$

Because  $s_0 = s_1$  is not settled yet, we are at the first iteration of the **while** loop.

## *f*-Bound Lemma: Proof (2)

Proof (continued).

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Because  $s_0 = s_1$  is not settled yet, we are at the first iteration of the **while** loop.

Because the state space is solvable and  $h$  is admissible, we have  $h(s_0) < \infty$ .

## *f*-Bound Lemma: Proof (2)

Proof (continued).

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Hence *open* contains the root  $n_0$ .

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Hence *open* contains the root  $n_0$ .

We obtain:  $f(n_0) = g(n_0) + h(s_0) = 0 + h(s_0) \leq h^*(s_0) = c^*$ , where “ $\leq$ ” uses the admissibility of  $h$ .

This concludes the proof for this case.

...

# *f*-Bound Lemma: Proof (3)

Proof (continued).

Case 2:  $i > 0$

Then  $s_{i-1}$  is settled and  $s_i$  is not settled.

Moreover,  $s_i$  is a solvable successor of  $s_{i-1}$  and  $\langle s_0, \dots, s_{i-1}, s_i \rangle$  is an optimal path from  $s_0$  to  $s_i$ .

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We can hence apply the optimal continuation lemma (with  $s = s_{i-1}$  and  $s' = s_i$ ) and obtain:

- (A)  $s_i$  is settled, or
- (B) *open* contains  $n'$  with  $n'.\text{state} = s_i$  and  $g(n') = g^*(s_i)$ .

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Because (A) is false, (B) must be true.

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(B) *open* contains  $n'$  with  $n'.\text{state} = s_i$  and  $g(n') = g^*(s_i)$ .

Because (A) is false, (B) must be true.

We conclude: *open* contains  $n'$  with

$f(n') = g(n') + h(s_i) = g^*(s_i) + h(s_i) \leq g^*(s_i) + h^*(s_i) = c^*$ ,  
where “ $\leq$ ” uses the admissibility of  $h$ .



# Optimality of $A^*$ with Reopening

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We can now show the main result of this chapter:

Theorem (optimality of  $A^*$  with reopening)

*$A^*$  with reopening is optimal when using an admissible heuristic.*

# Optimality of $A^*$ with Reopening: Proof

## Proof.

By contradiction: assume that the theorem is wrong.

Hence there is a state space with optimal solution cost  $c^*$  where  $A^*$  with reopening and an admissible heuristic returns a solution with cost  $c > c^*$ .

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This means that in the last iteration, the algorithm removes a node  $n$  with  $g(n) = c > c^*$  from *open*.

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With  $h(n.state) = 0$  (because  $h$  is admissible and hence goal-aware), this implies:

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With  $h(n.state) = 0$  (because  $h$  is admissible and hence goal-aware), this implies:

$$f(n) = g(n) + h(n.state) = g(n) + 0 = g(n) = c > c^*.$$

$A^*$  always removes a node  $n$  with minimal  $f$  value from *open*.

With  $f(n) > c^*$ , we get a contradiction to the  $f$ -bound lemma, which completes the proof. □

# Summary

# Summary

- $A^*$  with reopening using an admissible heuristic is optimal.
- The proof is based on the following lemmas that hold for solvable state spaces and admissible heuristics:
  - optimal continuation lemma: The open list always contains nodes that make progress towards an optimal solution.
  - $f$ -bound lemma: The minimum  $f$  value in the open list at the beginning of each  $A^*$  iteration is a lower bound on the optimal solution cost.